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Incompatibility, Modal Semantics, and Intrinsic Logic

1 Introduction

I closed Lecture 4 with an argument building on the idea that every autonomous discursive practice, in order to count as a *discursive* or *linguistic* practice, in order to count as deploying any *vocabulary*, must include performances that have the *pragmatic* significance of *assertions*, which on the *syntactic* side are utterances of *declarative sentences*, and whose *semantic* content consists of *propositions*. These pragmatic, syntactic, and semantic conditions form an indissoluble package, in the sense that one cannot properly understand any of the concepts assertion, sentence, and proposition apart from their relation to each other. This is the *iron triangle of discursiveness* (Figure 5.1).

I then proceeded to look at the pragmatic presuppositions of the assertional practices that are, on this account, PV-necessary to deploy any autonomous vocabulary. Here my claim was that no set of practices could count as according some performances the pragmatic significance of assertions unless it includes practices of giving and asking for *reasons*. That

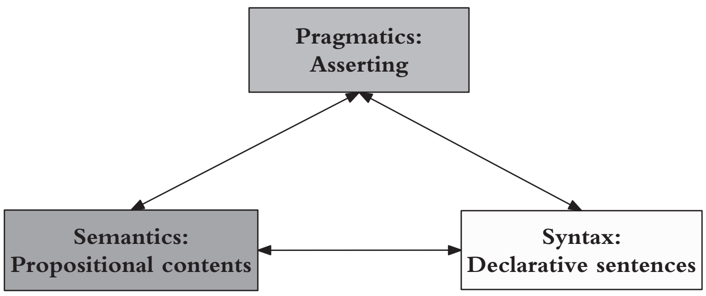

Figure 5.1 The iron triangle of discursiveness

is the claim that within the pragmatic dimension of the triad, *asserting* and *inferring* also form an indissoluble package, each element of which is in principle intelligible only in a context that includes the other. Assertional and inferential practices are reciprocally PP-necessary.

I then argued that any constellation of social practices is intelligible in principle as including the giving and asking for *reasons*—making *claims* whose status depends on their *inferential* relations to other claims that are their consequences, or have them as their consequences, or rule them out—only if it includes the capacity to distinguish two sorts of normative status as part of the pragmatic significance practically attributed to a speech act. To be giving and asking for *reasons*, interlocutors must practically distinguish (be able to respond differentially to) the sentences to which their interlocutors and they themselves are *committed* (based on those they are disposed to assert). And they must distinguish the sentences to which their interlocutors and they themselves are *entitled* (based on those they are committed to). These practical discriminative capacities need not be infallible (by any standard of ultimate correctness), and they need not be complete. But unless interlocutors make at least these two sorts of discrimination, what they are doing does not deserve to count as producing and consuming *reasons*, hence not as practically according some performances the pragmatic significance of *assertions*, hence not as deploying any autonomous *vocabulary*.[1]

My interest in the previous lecture was in arguing that these practices-or-abilities to discriminate commitments and entitlements are, in the terms of the sort of meaning-use analysis I have been developing here:

- PV-necessary for deploying any autonomous vocabulary,
- PP-sufficient by algorithmic elaboration for engaging in practices that are

[1] These are strong claims, no doubt contentious because tendentious: framed from the point of view of a normative pragmatist rationalism about the discursive. Those who are not convinced, those not tempted, and those not even willing to suspend disbelief on these points should just consider the remarks that follow as restricted to that sub-class of discursive practices that *does* exhibit the structure being considered—a sub-class that should at least be admitted to be large and significant, even by those who doubt that it plays the foundational and demarcational role here attributed to it. In any case, the principal arguments and constructions to be presented here as articulating incompatibility semantics do not depend on the particulars of the normative pragmatic metavocabulary in terms of which I want to understand incompatibility.

- PV-sufficient to deploy normative vocabulary, which is
- VP-sufficient to specify those original universally PV-necessary practices-or-abilities.

In sum, it was to argue that normative vocabulary—paradigmatically 'commitment' and 'entitlement'—stands in the complex resultant meaning-use relation of being elaborated-explicating (LX) with respect to every autonomous vocabulary. Whatever the status of that argument may be, my purpose here is to consider a different complex resultant meaning-use relation that the explicitly normative vocabulary of commitment and entitlement stands in to other vocabularies of philosophical interest, principal among them being alethic *modal* vocabulary. The relation I will focus on is that of one vocabulary's being a *pragmatic metavocabulary* for another. I want to explore a particular construction according to which normative vocabulary can serve as a pragmatic metavocabulary for *logical* vocabulary, including *modal* vocabulary, and how in those terms it can be seen to serve as such a metavocabulary for *semantic* vocabulary more generally. Along the way we will learn some lessons about logic and modality, and especially about the relation of *truth* and *compositionality* to semantics, that I think are of general interest, quite apart from the way in which they emerge from the particular pragmatic-analytic project I am pursuing here.

2 Incompatibility

The story I told about how engaging in practices of giving and asking for reasons requires the practical differential responsive ability to take or treat someone as committed and as entitled to the claims expressed by various sentences lets us make sense straightaway of two sorts of inferential relations between propositional contents on the semantic side, and corresponding practical dispositions on the pragmatic side. One takes or treats q as an inferential consequence of p in one sense by being disposed to attribute *commitment* to (what is expressed by) q to whomever one credits with commitment to (what is expressed by) p. And one takes or treats q as an inferential consequence of p in another sense by being disposed to attribute *entitlement* to the claim that q to whomever one credits with entitlement to the claim that *p*.[2] The first sort, commitment-preserving inferential relations, is a generalization, to include the case of non-logical, material inferences, of obligatory, *deductive* inferential relations. The second sort, entitlement-preserving inferential relations, is a generalization, to include the case of non-logical, material inferences, of permissive, *inductive* inferential relations. For example, anyone who is committed to a plane figure being rectangular is committed to its being polygonal. And the old nautical meteorological homily, ''Red sky at night, sailor's delight; red sky in morning, sailor take warning,'' tells us that anyone who sees a colorful sunrise is entitled to the claim that a storm that day is probable. But here the reasoning is only probative, not dispositive. The colorful sunrise provides *some* reason to predict a storm, but does not yet settle the matter. Other considerations, such as a rising barometer, may license one not to draw the conclusion one would otherwise be entitled to by the original evidence.

The abilities to take or treat interlocutors (including oneself) as committed or entitled to propositional contents expressed by various declarative sentences are PP-sufficient for the practical responsive recognition of another sort of semantic relation among propositional contents. This is because being disposed to respond to anyone who is *committed* to *p* as thereby precluded from counting as *entitled* to *q* (and vice versa) is treating *p* and *q* as *incompatible*. On the pragmatic side, this is a *normative* relation. It is not that one *can*not undertake incompatible commitments, make incompatible assertions. Finding that one has done so is an all-too-common occurrence. But the effect of doing so is to alter one's normative status: to undercut any entitlement one might otherwise have had to either of the incompatible commitments, for each commitment counts as a decisive reason against entitlement to the other, incompatible one.

On the pragmatic side, incompatibility can accordingly be thought of as a consequential relation like the other two:

- Incompatibility of *p* and *q*: If *S* is committed to *p*, then *S* is *not* entitled to *q*.
- Committive consequence: If *S* is committed to *p*, then *S* is committed to *q*.

[2] As will appear, entitlement-preserving inferences are always defeasible; the entitlement one acquires thereby is only *prima facie*. One is not entitled to the conclusion of a good entitlement-preserving inference if one is committed to something incompatible with it.

- Permissive consequence: If *S* is committed and entitled to *p*, then *S* is (*prima facie*) entitled to *q*.

But it is not immediately an *inferential* relation, since the conclusion is the *withholding* of a primary normative status, rather than the *inheritance* of one. Incompatibility relations *do*, however, underwrite a kind of inferential relation. The idea is an old one. Sextus Empiricus says, perhaps referring to Chrysippus:

> And those who introduce the notion of connexion say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent.[3]

My concern is not with when a *conditional* is sound, but with when the underlying *inference* that such a conditional is VP-sufficient to specify is a good one, in the material (that is, non-, or better, pre-logical) sense of "good inference" I am trying to articulate. And I do not want to assume at this stage that we are in a position to identify the *contradictory* of any claim. But the notion of material incompatibility can serve in its place. Making those adjustments yields the following definition:

> *p incompatibility-entails q* just in case everything incompatible with *q* is incompatible with *p*.

Thus "Pedro is a donkey," incompatibility-entails "Pedro is a mammal," for everything incompatible with Pedro's being a mammal (for instance, Pedro's being an invertebrate, an electronic apparatus, a prime number) is incompatible with Pedro's being a donkey.

I said before that the inferential relations among the propositional contents expressed by declarative sentences that correspond on the semantic side to inheritance of *commitment* can be thought of as a generalization (to the material case) of *deductive* inferential relations, and that those corresponding to inheritance of *entitlement* can be thought of as a generalization to the material case of *inductive* inferential relations. So we may ask: do *incompatibility*-entailments similarly generalize some kind of inferential relation that we already recognize in other terms? I think that they do, and that the inferences in question are counterfactual-supporting, *modally* robust inferential relations: the kind of inferences made explicit by *modally*

[3] Sextus, *Pyrrhoneiae Hypotyposes*, ii, 110–12, in William Kneale and Martha Kneale, *The Development of Logic* (Oxford University Press, 1962), 129.

qualified conditionals. The fact that the *properties* of being a donkey and being a mammal stand in the relation of incompatibility-entailment means that every *property* incompatible with being a mammal is incompatible with being a donkey. If two properties (such as being a mammal and being an invertebrate) are incompatible then it is *impossible* for any object simultaneously to exhibit both. And that means that it is *impossible* for anything to be a donkey and not be a mammal. That is why the incompatibility-entailment in question supports counterfactuals such as "If my first pet (in fact, let us suppose, a fish) *had been* a donkey, it *would have been* a mammal." We could say: "Necessarily, anything that is a donkey is a mammal."

On the semantic side, incompatibility is an implicitly *modal* notion. On the pragmatic side, the *normative* concepts of commitment and entitlement provide a pragmatic metavocabulary VP-sufficient to *specify* practices PV-sufficient to *deploy* that modal notion. That is, they let us *say* what it is one must *do* in order thereby to be taking or treating two claims *as* incompatible.[4] To begin to explore the consequences of this pragmatically mediated semantic relation between normative and modal vocabularies, we may consider the sort of grip on the semantics of expressions—the meanings expressed by deploying vocabularies—that one gets by thinking of their contents in terms of incompatibilities. I argued in Lecture 4 that there is an intimate connection between the conceptual contents expressed by vocabularies and the counterfactually robust inferences they are involved in. We might hope that a semantic metavocabulary centered on *incompatibility* would have the right expressive resources to make explicit important features of such contents. One case where we have particularly clear criteria of adequacy for our semantics is *logical* vocabulary. So I will be specifically concerned to offer an incompatibility semantics for logical vocabulary. Again, since incompatibility is at least implicitly itself a *modal* notion, we will want to see what an incompatibility semantics for modal vocabulary might look like. On this basis, one would hope to continue by elaborating a modal intensional semantics for *non*-logical vocabulary, as was done with possible worlds semantics in the second phase of the modal revolution.

[4] In my final lecture I shall be concerned to explore in much further detail this relation between what is expressed by modal and by normative vocabulary, as a way of thinking about the intentional nexus beween objects and the subjects who make claims about and act upon them.

3 Incompatibility semantics

Here is a semantic suggestion: represent the propositional content expressed by a sentence with the set of sentences that express propositions incompatible with it.[5] More generally, we can associate with each *set* of sentences, as its semantic interpretant, the set of *sets* of sentences that are incompatible with it.[6] The generalization from seeing incompatibility as a relation among sentences to seeing it as a relation among *sets* of sentences acknowledges an important structural fact about incompatibility: one claim can be incompatible with a set of other claims without being incompatible with any of its members. On the formal, logical side, where *incompatibility* is just *inconsistency*, p is incompatible with the set consisting of $p \rightarrow q$ and $\sim q$, but not with either individually. And on the side of non-logical content, the claim that the piece of fruit in my hand is a blackberry is incompatible with the *two* claims that it is red and that it is ripe, though not with either individually—in keeping with the childhood slogan that blackberries are red when they're green.

Aiming at maximal generality, I will impose only two conditions on the incompatibility relations whose suitability as semantic primitives I will be exploring here. First, I will only consider *symmetric* incompatibility relations. This is an intuitive condition because it is satisfied by familiar families of incompatible properties: colors, shapes, quantities, biological classifications, and so on. Second, if one set of claims is incompatible with another, so too is any larger set containing it. That is, one cannot remove or repair an incompatibility by throwing in some further claims. I call this the 'persistence' of incompatibility. If the fact that the monochromatic patch is blue is incompatible with its being red, then it is incompatible with its being red and triangular, or its being red *and* grass being green.

[5] Since, as has just been emphasized, incompatibility relations are only *one* dimension of inferential articulation, this semantic representation of conceptual content will necessarily be only partial.

[6] This generalization opens up a number of possibilities for correspondingly generalizing the incompatibility entailment relation. One very natural way to do that is to take it that a set of sentences X incompatibility entails a set Y just in case every set Z that is incompatible with Y is incompatible with X. In this case, $X \models \{y_1, \ldots, y_n\}$ has the meaning, X entails (y_1 and … and y_n). It turns out to be more formally convenient if instead one requires that X incompatibility entails Y in case every set Z incompatible with *every sentence in* Y is incompatible with X. In this case, $X \models \{y_1, \ldots, y_n\}$ has the meaning, X entails (y_1 or … or y_n).

Given any set of sentences, we can then define a standard *incompatibility interpretation* over that vocabulary as an incoherence partition of its power set that satisfies persistence. (Two sets of sentences are incompatible if and only if their union is incoherent.) Each such incompatibility interpretation induces an *incompatibility consequence* (or entailment) *relation* $\models$ in the way already indicated: being a cat entails being a mammal in this sense because every set of properties incompatible with being a mammal is also incompatible with being a cat.

The proposal here is to use incompatibility (itself introduced by a normative pragmatic metavocabulary) as the basic element of the *semantic metavocabulary*—and not just for *logical* expressions, but for ordinary *non*-logical vocabulary as well. The semantic interpretant of an object-vocabulary sentence is taken to be the set of sets of sentences materially incompatible with it.

The result is a *modal* semantics. For *incompatibility* is a *modal* notion. Now the development of modal semantic metavocabularies—in particular, the extension of possible world semantics from its initial home as a semantics for modal *logical* vocabulary to a modal semantics for ordinary, non-logical expressions in general—is perhaps the principal technical philosophical advance of the past forty years.[7] (It is the second of three sequential modal revolutions in recent philosophy—or of three phases of one complex, multi-stage revolution—the first being Kripke's formal possible worlds semantics for modal *logic*, and the third beginning with his application of that apparatus to the semantics of proper names.) I want to take that hint, but to apply modal vocabulary to semantic projects in a somewhat different way: using the notion of incompatibility to provide a *directly* modal semantics. By that I mean one that does not approach modality by beginning with a more basic semantic notion of *truth*.

Classical possible-worlds semantics proceeds in two stages. Like more traditional semantics, its basic semantic notion is that of truth. It begins by relativizing evaluations of truth to points of evaluation—paradigmatically, possible worlds. Then, at the second stage, necessity and possibility can be introduced by quantification over such points of truth-evaluation—possibly exploiting structural relations among them, such as accessibility relations

[7] For an example, one can consider the story about the semantic distinction between attributive and non-attributive adverbs that I tell in the Afterword.

among possible worlds, or the ordering of time and place co-ordinates. The semantic interpretants of expressions are in the first instance functions from points of evaluation to extensions or truth-values. This is one natural way to capture the element of *generality* that Ryle insisted was present in all endorsements of inferences:

> ... some kind of openness, variableness, or satisfiability characterizes all hypothetical statements alike, whether they are recognized "variable hypotheticals" like "For all *x*, if *x* is a man, *x* is mortal" or are highly determinate hypotheticals like "If today is Monday, tomorrow is Tuesday."[8]

By contrast to such two-stage approaches, semantics done in terms of incompatibility is *directly* modal. One may, if one likes, think of the incompatibility of *p* and *q* as the impossibility of both being *true*. But that characterization in terms of truth is entirely optional. Incompatibility is itself already a modal notion, and for semantic purposes we can treat it as primitive. The explication I have offered is in pragmatic terms: *saying* (in terms of the normative notions of commitment and entitlement) what one must *do* in order to be taking or treating two claims *as* incompatible. The element of generality comes in because in assessing entailments we look at *all* the claims that are incompatible with the conclusions and the premises. One claim is an incompatibility consequence of another only if there is *no* set of sentences incompatible with the conclusion and not with the premises. And here it is important that the potential defeasors are *not* limited to sentences that are *true*. Even if as a matter of fact all the coins in my pocket are copper, that a coin is in my pocket does not *entail* that it is copper, since "This coin is silver" is incompatible with its being copper, but *not* with its being in my pocket, even though it is not *true* that it is in my pocket. For, as we want to say, it *could* be in my pocket: that non-actual state of affairs is *possible*. That modal fact is reflected in the fact that a coin's being silver is not *incompatible* with its being in my pocket. The idea that I want to explore is that once we have properly learned the lesson that modality matters in semantics because counterfactually robust inferences are an essential aspect of the articulation of the conceptual contents of sentences, the way is opened up to a *directly* modal semantics, which does *not* make what now appears as an unnecessary preliminary *detour* through assessments of *truth*.

[8] Gilbert Ryle " 'If', 'So', and 'Because'," in Max Black (ed.), *Philosophical Analysis* (Prentice Hall, 1950), 302–18, at 311.

This is all very abstract. In order to see incompatibility semantics in action, we should look to the case where the criteria of adequacy of a semantics are clearest: namely, to semantics for *logical* vocabulary. That, after all, is where possible-worlds semantics cut its teeth.

4 Introducing logical operators

The notion of incompatibility can be thought of as a sort of conceptual vector-product of a *negative* component and a *modal* component. It is *non-compossibility*. To use this semantic notion to introduce a negation operator into the object vocabulary, we must somehow isolate and express explicitly that negative component. The general semantic model we are working with represents the content expressed by a sentence by the set of sets of sentences incompatible with it. So what we are looking for is a way of computing what is incompatible with negated sentences (and, more generally, with sets of sentences containing them). Since we do not have any sort of yes/no evaluation of sentences in the picture (not even a relativized one), we cannot approach negation as a kind of reversal of semantic polarity. How else might we think about it?

Incompatible sentences are Aristotelian *contraries*. A sentence and its negation are *contradictories*. What is the relation between these? Well, the contradictory is *a* contrary: any sentence is incompatible with its negation. What distinguishes the contradictory of a sentence from all the rest of its contraries? The contradictory is the *minimal* contrary: the one that is entailed by all the rest. Thus every contrary of "Plane figure *f* is a circle"—for instance "*f* is a triangle," "*f* is an octagon," and so on—entails "*f* is *not* a circle." *Blue, green, yellow* all entail *not-red*. For any sentence *p* we are assuming that we can already pick out its contraries, that is, the (sets of) sentences that are incompatible with it. And we already have an entailment relation, defined wholly in terms of incompatibility. So we have all the resources needed to say that some other sentence *q* is the negation of *p* just in case *q* is the *minimal* incompatible of *p*: the one entailed by everything else incompatible with it.

It might happen that in some standard interpretation of the vocabulary to which *p* belongs, there already is such a *q*. But we cannot count on every sentence already having such a negation in every interpretation. So we need to introduce new sentences, of the form N*p*, on the basis of this relation. Inspection of the definition of incompatibility entailment yields the result that N*p* will be an inferentially minimal incompatible of *p* if and only if a set of sentences is incompatible with it just in case that set entails *p*. This is equivalent to saying that what is incompatible with the negation of *p* is what is incompatible with *every* set of sentences incompatible with *p*—that is, that the incompatibility set of N*p* is just the *intersection* of the incompatibility sets of everything incompatible with *p*.[9]

This definition lets us recursively add, for every sentence *p* of the language, its negation N*p*, and to compute the incompatibility sets of those negations so as to satisfy the principle that everything incompatible with *p* entails N*p*. Extending the incompatibility relation to apply to sets of sentences that include arbitrarily iterated negations automatically extends the incompatibility consequence relation, which is defined in terms of it. And it is easy to show that that extension is *inferentially conservative*—that is, that the extended consequence relation does not add or subtract any consequences that involve only the old vocabulary.

What are the properties of negation, given this incompatibility semantics? It turns out to have all the familiar and desirable properties we expect in a negation:

- Because *p* and N*p* are guaranteed to be incompatible, every set of sentences that contains or entails both—what we are now in a position to characterize as the *inconsistent* sets of sentences—is guaranteed to be incoherent.
- Negation contraposes appropriately with incompatibility entailment. That is, *p* entails *q* if and only if *not-q* entails *not-p*.
- And every sentence is incompatibility-equivalent to its double negation: $p \models NNp$ and $NNp \models p$.

Further logical properties of negation depend on its interaction with other connectives, and accordingly must be considered after we have introduced them.

So the procedure is to start with a *material* incompatibility-and-consequence structure that articulates the contents of *non*-logical vocabulary, and on that basis introduce *logical* vocabulary—in this

[9] One may consult the appendices to this lecture to see how this intuition is extended to the case of sets of sentences containing negations.

case negation—whose content is derived from that of the non-logical vocabulary on which it is based. A corresponding procedure permits the introduction of *conjunction*. Here the most important fact to acknowledge is that something can be incompatible with a conjunction even though it is not incompatible with either conjunct. That the fruit in my hand is a blackberry is incompatible with its being red *and* ripe, even though it is *not* incompatible with either one individually. This is the phenomenon that led us to think about incompatibility relations among *sets* in the first place. And that is the clue as to how to compute the incompatibilities of conjunctions. What is incompatible with the conjunction K*pq* should just be whatever is incompatible with the *set* $\{p, q\}$. Once again we can introduce conjunctions recursively and conservatively in this way, along with negations, so as to extend any standard incompatibility relation by computing incompatibilities for all sentences formed from basic vocabulary of primitive proposition letters by arbitrary iterations of conjunction and negation.

It is easy to show that under this definition, conjunction acts like conjunction. It is obvious from the semantic definition that the set consisting of the two premises p and q entails their conjunction ($p, q \models \mathrm{K}pq$), and it follows immediately from the persistence of incompatibility that a conjunction entails each of the conjuncts ($\mathrm{K}pq \models p$ and $\mathrm{K}pq \models q$). It is less obvious, but turns out to be true, that this definition also validates the principle that if p entails q and p entails r, then p entails their conjunction.

In fact, conjunction behaves classically. Furthermore, it interacts with negation in the familiar ways: the full logic is distributive. To make a long story short, the logic generated by these semantic definitions of negation and conjunction in terms of incompatibility is just classical logic. Notice that negation and conjunction are not interpreted semantically by anything at all like *truth*-functions. As we'll see, their semantics is in both cases *intensional* in *definition*, but nonetheless *extensional* in *result*—in the sense that it yields just the theorems of classical two-valued logic.

What sets incompatibility semantics apart, however, is that we can exploit the fact that *incompatibility* is a *modal* semantic primitive to introduce *modal* logical vocabulary in the very same setting, and the very same terms, in which we introduce the classical *non*-modal logical vocabulary.

On the semantic approach I am pursuing, to introduce a connective one specifies how to compute its incompatibilities. So the question is: what intuitively should be taken to be *incompatible* with *necessarily-p*, that is, with the *necessity* of *p*? Put otherwise, what claims rule out the *necessity* of *p*? Clearly, anything incompatible with *p* is incompatible with *necessarily-p*. Given the definition of entailment, this just says that the rules for computing the incompatibilities of *necessarily-p* should ensure that *necessarily-p* entails *p*. But what else is incompatible with the *necessity* of *p*, besides the things that are incompatible with *p*? Here is the basic thought. **To be incompatible with *necessarily-p* is to be** (self-incompatible or) **compatible with something that *does not entail p***. For anything compatible with something that does not entail *p* is compatible with something that does not necessitate *p*, and so leaves open the possibility that *p* is not necessary.

A similar line of thought applies to *possibility* in relation to incompatibility, permitting us to introduce *possibly-p* as well as *necessarily-p*. Whatever is incompatible with *possibly-p* should be incompatible with *p* (ensuring that *p* entails *possibly-p*). But only some things that rule out *p* also rule out the *possibility* of *p*. Which are those? Here is an idea. **To be incompatible with *possibly-p* is to be incompatible with *everything* that is *compatible* with something *compatible* with *p***. For anything compatible with something compatible with *p* is compatible with something that leaves the *possibility* of *p* open. It turns out to be straightforward to show that, according to these definitions, *possibly-p* is incompatibility-equivalent to *not-necessarily-not-p*, and *necessarily-p* is incompatibility-equivalent to *not-possibly-not-p*, given the way we have defined negation above. So these definitions fit together in the way we would expect.

To make another long story short, the modal-logical theorems that are valid on all standard incompatibility frames given these definitions are just those of the familiar Lewis system S5. This is the system in which it is true both that whatever is *necessary* is *necessarily* necessary and that whatever is *possible* is *necessarily* possible. In the usual Kripke semantics, this is the modal logic generated by accessibility relations among possible worlds that are reflexive, symmetric, and transitive. In the tangled jungle of modal-logical systems, this is the unexciting, well-studied, well-behaved, plain-vanilla modal analogue of the classical non-modal propositional calculus.

5 Meaning-use analysis

Figure 5.2 shows a meaning-use diagram corresponding to this incompatibility semantics for modal logical vocabulary.

Here is some help in reading it:

- Basic meaning-use relations (MURs) 1–3 are by now familiar. I have argued that every autonomous discursive practice must include practices of giving and asking for reasons—as part of the iron triangle of discursiveness—and that that involves distinguishing in practice between the deontic statuses of commitment and entitlement.
- We saw last time that that is sufficient to introduce normative vocabulary, specifically the deontic vocabulary of 'commitment' and 'entitlement', which is VP-sufficient to specify the triadic inferential

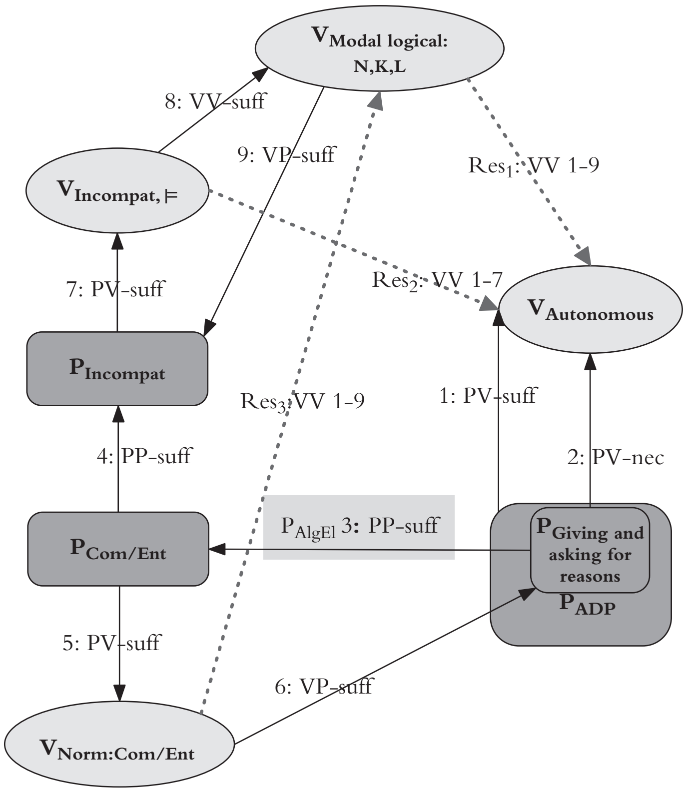

Figure 5.2 Incompatibility semantics for modal logic

substructure of practices of giving and asking for reasons. Those facts are represented by MURs 5 and 6.

- We also saw how practically distinguishing commitments and entitlements underwrites a notion of practical incompatibility of commitments, where commitment to one claim is taken or treated as sufficient to rule out entitlement to another. That is MUR 4, which permits the introduction of a semantic metavocabulary that lets one say that two claims are incompatible, and that claims stand in the relation of incompatibility-entailment, which is MUR 7.
- We have now seen how that semantic metavocabulary allows one to extend the original vocabulary by introducing modal-logical vocabulary (MUR 8), which has the expressive power to define a connective that says in that object-vocabulary *that* two claims are incompatible: LNK*pq*. Basic MUR 9 accordingly exhibits modal-logical vocabulary as a kind of *semantic* metavocabulary for incompatibility.
- Complex resultant MUR Res_1 analyzes the sense in which the vocabulary of modal logic S5 is *implicit* in the use of any autonomous vocabulary. This analysis is a further cashing-out of what last time I called "The modal Kant-Sellars thesis."
- Complex resultant MUR Res_2 codifies an analysis of the possibility of using <u>incompatibility</u> and <u>incompatibility-entailment</u> as a semantic metavocabulary for any autonomous vocabulary.
- Finally, complex resultant MUR Res_3 represents a new relation between the *normative* vocabulary of commitment and entitlement and the *modal* vocabulary of necessity and possibility. It represents a detailed analysis of a sense in which we could understand Sellars's dictum that "the language of modality is a 'transposed' language of norms." When I introduced that slogan in Lecture 4, I suggested that the way to fill in Sellars's black-box notion of 'transposition' was in terms of the pragmatically mediated semantic relation of providing a *pragmatic* metavocabulary. I offered a simple MUD as a representation of this relation (repeated here as Figure 5.3).

We are now in a position to unpack what were there represented as two basic MURs. The PV-sufficiency of a set of modal practices for the deployment of a modal vocabulary in this simple diagram corresponds to the complex MUR that is the resultant of basic MURs 7 and 8 in the MUD

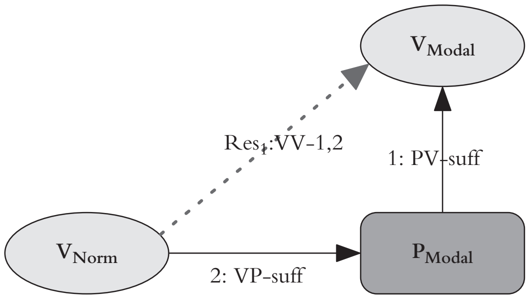

Figure 5.3 "The language of modalities is a 'transposed' language of norms."

for modal logic (Figure 5.2). The VP-sufficiency of a normative vocabulary to specify those implicitly modal vocabulary-deploying practices now shows up as the complex MUR that is the resultant of relations 3, 4, 5, and 6. In fact, we ought to include the other basic MURs that occur in our diagram, and identify the resultant representing the fact that normative vocabulary can serve as a pragmatic metavocabulary for modal vocabulary in the simple MUD with the complex resultant relation 3 in the more complex MUD.

The MUD of Figure 5.2 accordingly offers a detailed analytic reading of the Sellarsian claim that "the language of modality is a 'transposed' language of norms," understood as asserting a complex pragmatically mediated semantic relation between deontic and alethic modal vocabularies. Now, Sellars's claim might or might not be correct. And this interpretation-as-analysis of it might or might not be correct. But I take it to be a signal measure of the power of the metaconceptual apparatus of meaning-use analysis that it so much as permits the expression of this detailed a reading. And I have worked hard here to justify MURs 4, 7, 8, and 9. Regardless of how successful those efforts have been, the fact that the meaning-use analysis tells us *exactly* what constellation of sub-claims we must argue for in order to justify the overall account seems to me to constitute concrete progress in our grasp of and control over the philosophical claims we make in this area.

The complex pragmatically mediated semantic relation between deontic and alethic modal vocabularies that shows up here indicates that there is a deep relation between what in the previous lecture I called the "*modal* and *normative* Kant-Sellars theses." In the final lecture I will have more to say about this relation, and about what it has to do with what is expressed by *intentional* vocabulary.

6 Semantic holism: recursive projectibility without compositionality

Returning closer to ground-level, however, there are two more lessons I take to be of some potential philosophical significance that can be drawn from the construction of an incompatibility semantics for modal-logical vocabulary. The first concerns debates about semantic *holism* and *compositionality*.

As with the familiar Kripke semantics for modal vocabulary, the meta-vocabulary in which incompatibility semantics is conducted is entirely *extensional*. The semantic interpretants of sentences (and theories) are just sets (of sets of sentences), and the semantic interpretants of logically compound sentences are computed by purely set-theoretic operations on those sets. Also as with the Kripke semantics, this is possible because an overtly modal semantic primitive is appealed to: in the one case *accessible possible world*, in the other case *incompatibility*. (A significant difference is that I have offered a normative, deontic, pragmatic metavocabulary in which to *say* what you have to *do* to deploy that modal semantic primitive, and hence, eventually, the modal operators semantically defined in terms of it.)

The operators defined by the extensional incompatibility semantics are strongly *intensional*, however. We have noticed that one cannot in general compute the incompatibilities of a conjunction from the incompatibilities of its conjuncts. For something can be incompatible with a conjunction without being incompatible with either of its conjuncts. And things are much worse with negation. The two commitments:

- to defining *p* as incompatibility-entailing *q* just in case everything incompatible with *q* is incompatible with *p*, and
- to understanding the negation of *p* as its inferentially weakest incompatible, that is, as what is incompatibility-entailed by everything incompatible with *p*,

together have as a consequence that, to be incompatible with *not-p* is just to be in the intersection of the incompatibility-sets of everything incompatible with *p*. But that means that we can hold fixed what is incompatible with *p*, and by varying the incompatibility-sets of some of *those* elements alter the incompatibility-set of *not-p*. It follows that in each incompatibility-interpretation, the semantic value of *not-p* is *not* determined by the semantic value of *p* alone, but only by it together with the semantic values of a *lot* of other sentences not mentioned in the formula—namely those incompatible with those incompatible with *p*.

It is perhaps less surprising that the incompatibility definitions of what is expressed by necessity and possibility are also intensional, in much the same sense that negation is. So for instance, what is incompatible with *possibly-p* is what is *in*compatible with everything *compatible* with something *compatible* with *p*. Once again, we can fix the semantic interpretant of *p*, its incompatibility set, and still vary the semantic interpretant of *possibly-p*, by varying the semantic interpretants of things compatible with what is compatible with *p*. And the same phenomenon is exhibited by the incompatibility definition of *necessarily-p*.

This is to say that the classical and modal-logical connectives, as semantically defined by incompatibilities, do not have the *semantic sub-formula property*. That is, it is *not* the case that the semantic interpretants of logical compounds formed by applying those connectives is a function of the semantic interpretants of their components, to which the connectives are applied. It is *not* possible to compute the semantic values of arbitrary logical compounds of primitive sentences just from the semantic values of the sentences and the connectives from which they are formed. Another way to put this point is that the incompatibility semantics for these connectives is *not compositional*. It is in this precise sense a *holistic* semantics, in that what is incompatible with (and hence an element of the semantic value of) *not-p* or *necessarily-p* or *possibly-p* depends on what is incompatible with (and hence on the semantic value of) other sentences *q* linked with *p* in that they are compatible or incompatible with it, or incompatible with something that entails it, or compatible with something compatible with it. The holistic character of incompatibility semantics—whether for logical expressions such as 'not' or material, non-logical ones such as 'triangular'—is a result of its codifying the so-to-speak *horizontal* dimension of semantic content, the one that is articulated by the relations of sentences to each other, rather than the *vertical* dimension, which consists in their relations to things that are not themselves sentences.

It is widely believed, and has been particularly forcefully argued by Jerry Fodor, that no holistic semantics can account either for the *projectibility* of language or for its *systematicity*, and hence not for its *learnability*. That is, it is argued that *only* on the assumption that semantics is *compositional* can we account for the determinateness of the semantic values of an indefinite number of novel compounds of simple expressions, for the fact that wherever some syntactic combinations of those simple expressions have semantic values so do others systematically related to them, and for the fact that speakers of a language can produce and understand an indefinite number of novel compounds, systematically related to one another by their modes of formation, upon mastering the use of the simple expressions and modes of formation.

But I think we are now in a position to see that those arguments cannot be right. They depend upon systematically overlooking the possibility of semantic theories that have the shape of the incompatibility semantics for classical and modal-logical vocabulary we have been considering. For—and this is the key point—although that semantics is *not compositional*, it *is* fully *recursive*. The semantic values of logically compound expressions are wholly determined by the semantic values of logically simpler ones. It is *holistic*, that is, *non*-compositional, in that the semantic value of a compound is *not* computable from the semantic values of its components. But this holism *within* each level of constructional complexity is entirely compatible with recursiveness *between* levels. And this is not just a philosophical *claim* of mine. The system I am describing allows us to *prove* it. (In this context, proof is the word made flesh.)

The semantic values of all the logically compound sentences are computable entirely from the semantic values of *less complex* sentences. It is just that one may need to look at the values of *many*—in the limit *all*—the less complex sentences, not just the ones that appear as sub-formulae of the compound whose semantic value is being computed. The semantics is *projectible* and *systematic*, in that semantic values are determined for all syntactically admissible compounds, of arbitrary degrees of complexity. It is *learnable*—at least in principle, putting issues of contingent psychology aside, in the ideal sense we have been working with. For the capacity to distinguish the incompatibility-sets of primitive propositions is, in the context of the semantic definitions of the connectives in terms of incompatibilities I have offered, *PP-sufficient by algorithmic elaboration* for the capacity to distinguish the incompatibilities of all their logical (including modal-logical) compounds—and hence for the practical capacity to distinguish what is a consequence of what.

What semantic *projectibility, systematicity*, and *learnability-in-principle* require, then, is not semantic *atomism* and *compositionality*, but semantic *recursiveness* with respect to complexity. That is entirely compatible with the semantics being *holistic*, in the sense of *lacking* the *semantic sub-formula property*, which is the hallmark of atomism and compositionality. And the argument for this claim is not merely the description of an abstract possibility. The incompatibility semantics for logical vocabulary provides an up-and-running counterexample to the implicit assumption that semantic recursiveness is achievable only by compositionality. Having compound expressions exhibit the semantic sub-formula property is only *one* way of securing recursiveness. The standard arguments for semantic compositionality are fallacious.[10]

So here is another side-benefit of or philosophical spin-off from looking analytically at *pragmatically mediated* semantic relations between antecedently philosophically interesting vocabularies—and a valuable lesson we can learn from the incompatibility semantics that arose from thinking about complex resultant meaning-use relations between normative and modal vocabularies.

7 Consequence-intrinsic logic

The order of explanation I have been pursuing up to this point,

- starts with practices of giving and asking for reasons,
- argues that they are PP-sufficient for practices of deploying basic *normative* vocabulary—in particular the deontic modal vocabulary of 'commitment' and 'entitlement',
- uses that as a pragmatic metavocabulary that specifies how to deploy a *modal* concept of <u>incompatibility</u>,

[10] A more charitable way to put things would be that compositionality—which really amounts to semantic recursiveness—has been confused with the semantic sub-formula property. Thus Jerry Fodor and Ernest Lepore's *The Compositionality Papers* (Oxford University Press, 2002) opens with this definition: "Compositionality is the property that a system of representations has when (i) it contains both primitive symbols and symbols that are syntactically and semantically complex; and (ii) the latter inherit their syntactic/semantic properties from the former" (p.1). On this definition, the incompatibility semantics is fully compositional. But Fodor and Lepore go on to draw atomistic consequences from compositionality that in fact only follow from the semantic sub-formula property. So the confusion I am concerned to point out is in play, however we decide to specify it.

- uses that as the basic semantic metavocabulary in which to define a *consequence* relation of incompatibility-entailment,
- and on that basis offers semantic definitions of logical vocabulary, including modal operators.

It is possible to exploit the pragmatic and semantic relations appealed to in this approach in service of a *different*, converse, order of explanation, however. In particular, instead of defining a semantic consequence relation in terms of a *prior* notion of incompatibility, we can *start* with a *consequence* relation—either a *logical* consequence relation or a *material* one that depends on the contents of the *non*-logical vocabulary articulating its premises and conclusions—and *impute* an incompatibility relation on that basis so as semantically to generate just that consequence relation by the procedures I have already put in place.

The idea is to hold fixed the principle that Y is a semantic consequence of X just in case everything incompatible with Y is incompatible with X, but to use that principle relating them to define an incompatibility relation among sets of sentences of the language that would generate whatever consequence relation we are given to begin with. To make this work, we have to ask what conditions a consequence relation defined on an arbitrary set of sentences must meet in order to make it possible to define from it an incompatibility relation such that sets of sentences X and Y stand in the consequence relation (which I'll write '$X \vdash Y$') just in case everything incompatible with Y is incompatible with X (which I will continue to write '$X \models Y$').

It turns out that two conditions suffice:

1. General Transitivity: $\forall X, Y, Z, W \subseteq L[(X \vdash Y \;\&\; Y \cup W \vdash Z) \rightarrow X \cup W \vdash Z]$.
2. Defeasibility: $\forall X, Y \subseteq L[\sim(X \vdash Y) \rightarrow \exists Z \subseteq L[\forall W \subseteq L[Y \cup Z \vdash W] \;\&\; \exists W \subseteq L[\sim(X \cup Z \vdash W)]]]$.

I will call any consequence relation meeting these conditions 'standard'. The first is a very minimal condition on consequence relations, which corresponds to the usual 'Cut' rule of sequent calculi:

$$\frac{\Gamma : A \text{ and } \Delta, A : B}{\Gamma, \Delta : B}$$

The second says that if Y is *not* a consequence of X, then there is something that yields an absurdity—something that has *everything* as a consequence—when added to Y but not when added to X. In Appendix 4 to this lecture, I show that if a consequence relation meets these two conditions, then it is possible to define an incompatibility relation that will generate exactly that consequence relation as incompatibility-entailment, by identifying incoherent sets as those that have all sets as their consequences, and then taking two sets to be incompatible if and only if their union is incoherent. That is, for every standard consequence relation, we can find a standard incompatibility relation that semantically validates it.

How interesting the representation theorem that comprises these soundness-and-completeness results is depends on how severe a constraint on consequence relations that second condition is. The easiest way to assess that is to see what familiar, or otherwise interesting, consequence relations do and do not satisfy it. I have already argued, in effect, that sentences attributing ordinary compatible families of incompatible properties—paradigmatically, shapes and colors, and membership in various biological or physical kinds, but encompassing a great many others as well—exhibit material consequence relations that are *standard* in this sense. This is because those sentences stand in the material consequence relations that are defined by their incompatibilities, and those, the representation theorem shows, meet the two conditions of standardness. But what of others, which are not defined to begin with in terms of incompatibility? The consequence relations we understand best are *logical* consequence relations, defined on logical vocabulary by various sets of axioms concerning derivability. Perhaps at this point it comes as no surprise that the consequence relation characteristic of classical logic is a standard one. In that setting, a set of sentences can be taken to be *incoherent* just in case it is *inconsistent*, in that for some p, both p and $\sim p$ can be derived from it. Any superset of an inconsistent set is inconsistent, and from inconsistent sets one can derive everything. Treating two sets as *incompatible* just in case their union is *inconsistent* then yields the classical consequence relation under our usual definition of semantic incompatibility-entailment. And this result holds for the first-order quantificational calculus just as it does for the classical propositional calculus. The consequence relations of both logics are standard, and so can be completely codified semantically by incompatibility relations.

Logically inconsistent sets play just the same role in the consequence relations of standard Lewis modal systems, such as S4 and S5, as they do in the classical logic on which they are based. So these too are standard, and so semantically codifiable as incompatibility-entailment relations.

Now, the observation with which I shall close is this: our representation theorem shows that any consequence relation that meets the conditions of standardness—whether it be a material or a logical consequence relation—can be codified by a standard incompatibility relation definable in a natural way from that consequence relation. And we have seen that any standard incompatibility relation has a logic whose non-modal vocabulary behaves classically and whose modal vocabulary is S5, in the sense that the natural semantic definitions of such vocabulary in terms of incompatibility yields that logic. Putting these results together, we can say that in this precise sense, S5 (whose non-modal fragment is just classical logic) is the logic *intrinsic to* standard incompatibility relations, and hence standard consequence relations. But since not only classical logic, but *all* the usual modal logics—not only S5, but K, T, S3, S4, and B, have standard consequence relations, classical logic and S5 are the intrinsic logic of, for instance, S4, as well as the others. And although the consequence relation of intuitionistic logic is not standard, so not codifiable by a standard incompatibility relation, in a natural sense it does implicitly contain a standard consequence relation, and so in this somewhat extended sense it, too, has PC + S5 as its intrinsic logic.[11] And in the same sense, so does intuitionistic S4. Relevance logic aside, the logic that is in this sense *intrinsic* to the consequence relations of *most* other familiar logics is classical S5. **S5 accordingly has some claim to being *the* modal logic of consequence relations, whether material or logical**.

The concept of the logic that is *intrinsic* to the consequence relation characteristic of some vocabulary (whether logical or not) is the concept of a new kind of semantic relation between vocabularies. It is mediated by the vocabulary of incompatibility, in terms of which, on the one hand, the consequence relation is codified.

I have been concerned to fill in the three sets of practices that implement the basic VV-sufficiency relations of which the relation of *intrinsicness* of a logic to a vocabulary is the resultant:

- imputing a standard *incompatibility* relation from a standard *consequence* relation (P1 of Figure 5.4);

[11] I discuss this point further in Appendix 4.2.3 below.

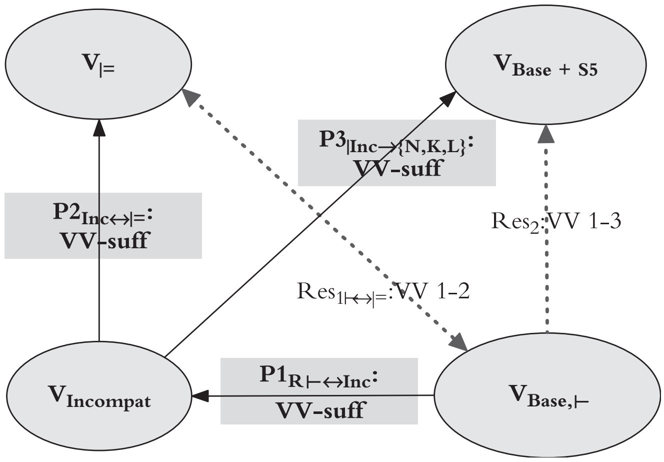

Figure 5.4 Consequence-intrinsic logic

- defining incompatibility-entailment in terms of that incompatibility relation (P2); and
- semantically introducing logical vocabulary, including modal vocabulary, in terms of incompatibility (P3).

So here is another payoff from the metaconceptual apparatus of meaning-use analysis—one that played no part in its initial motivation. The complex resultant MUR this constellation of basic MURs defines is a semantic relation that, apart from this methodology, we would never have been in a position to notice: the relation between logical vocabularies and other vocabularies, when the logical vocabulary is *intrinsic* to the consequence relation *characteristic* of the other vocabulary.

Having put the technical material behind us, in the final lecture I will shift focus by turning attention to what is expressed by *intentional* vocabulary and take up once again, from yet a different perspective, the issue of the relations between normative and modal vocabularies—or if you prefer, between deontic and alethic modalities—as it bears on the nature of intentionality, which itself shows up in this context as itself a *pragmatically mediated semantic relation*.

Technical appendices[12]

Appendix 1: Incompatibility semantics

1 *Definitions and axioms*

1.1 Incoherence, incompatibility, and entailment We are given a language L, which is a set of sentences. L may be purely atomic, or it may contain logically complex formulae. L is *proper* if, for each $p \in L$ and each q a subformula of p, we also have $q \in L$. All languages under consideration are proper.

Axiom (Persistence) For finite $X, Y \subseteq L$, and $X \subseteq Y$, if $X \in \text{Inc}$ then $Y \in \text{Inc}$.

Incoherence is a generalization of *inconsistency* to the case of non-logical properties. Persistence says that if a set is incoherent, adding more sentences to it cannot cure that condition. An ordered pair $< L, \text{Inc} >$ whose second element is an incoherence property defined over the first element is a *standard incoherence frame* on L. (Henceforth all frames are understood to be standard—that is, their incoherence property satisfies Persistence.)

Incompatibility An *incompatibility function* I is a function from $P(L)$ to $P(P(L))$.

Incoherence ~~relations~~ properties are related one-to-one to incompatibility functions by:

Axiom (Partition) $X \cup Y \in \text{Inc}$ iff $X \in \text{I}(Y)$.

[12] The formal work presented in these appendices is the result of a collaboration with my research assistant (and Pitt PhD student) Alp Aker. I came up originally with versions of the semantic definitions, the introduction rules for the connectives, proofs of the validity of the various logical principles involving those connectives, and most of the other results reported in these appendices. Alp vastly improved our understanding of the incompatibility semantics by shifting to a definition of incompatibility entailment that is disjunctive on the right (I had used one that was conjunctive on the right). This made it possible for him to formulate the reduction formulae for the connectives, which made all the proofs cleaner and easier. (I had been working directly from the basic definitions, which required extremely laborious derivations from very quantificationally complex formulae.) It also made it possible for him to prove the crucial metatheorems showing that the semantic connective definitions determine extensions of incompatibility frames over a set of non-logical sentences to arbitrarily complex logical compounds of them in a way that is inferentially conservative and unique. (The explicit recursions I had attempted to use to the same end proved unworkable.) Alp is also responsible for the proofs of completeness, soundness, and compactness in Appendix 3.

That is, two sets of sentences are incompatible just in case their union is incoherent. An ordered pair $< L, \mathrm{I} >$ is a *standard incompatibility frame* on L. Note that incompatibility is symmetric: if $X \in \mathrm{I}(Y)$, then $Y \in \mathrm{I}(X)$. It also follows from the Persistence of incoherence that if $X \in \mathrm{I}(Y)$ and $X \subseteq X'$, then $X' \subseteq \mathrm{I}(Y)$.

Given an incoherence relation or an incompatibility function we have the following relation of *incompatibility-entailment*:

Entailment $X \models_{\mathrm{I}} Y$ iff $\bigcap_{p \in Y} \mathrm{I}(\{p\}) \subseteq \mathrm{I}(X)$.

X can be an infinite set of formulae, but *we require Y to be finite*. When Y is empty we read $\bigcap_{p \in Y} \mathrm{I}(\{p\})$ as equivalent to $P(L)$. (Thus $X \models_{\mathrm{I}} \emptyset$ is equivalent to $X \in$ Inc.) We index entailment relations by incompatibility functions (or, equivalently, by incoherence properties). The underlying idea is that one sentence incompatibility-entails another if and only if everything incompatible with the conclusion is incompatible with the premise. That idea is generalized to a relation between sets in a convenient and natural way. The heuristic meaning of $X \models \{y_1, \ldots, y_n\}$ is that X entails y_1 *or* … *or* y_n.

Validity X is *valid* iff $Y \in \bigcap_{p \in X} \mathrm{I}(\{p\}) \Rightarrow Y \in$ Inc.

Thus $\models \{p\}$ is equivalent to $\{p\}$'s being valid.

1.2 Connective definitions We have three axioms that govern the behavior of the connectives N, K, and L (which are introduced, and their definitions motivated, in the body of the text):

Axiom (Negation Introduction; NI) $X \cup \{\mathrm{N}p\} \in$ Inc iff $X \models \{p\}$.

Axiom (Conjunction Introduction; CI) $X \cup \{\mathrm{K}pq\} \in$ Inc iff $X \cup \{p, q\} \in$ *Inc*.

Axiom (L Introduction; LI) $X \cup \{\mathrm{L}p\} \in$ Inc iff $X \in$ Inc or $\exists Y[X \cup Y \notin \text{Inc} \;\&\; Y \not\models \{p\}]$.

For ease of reading we sometimes drop brackets around sets and sometimes use the comma to denote set union. Thus we can, for example, write $\mathrm{I}(p, X)$ instead of $\mathrm{I}(\{p\} \cup X)$ and $X, \mathrm{N}p \models q$ instead of $X \cup \{\mathrm{N}p\} \models \{q\}$. We also write A$pq$ as an abbreviation of NKNpNq, Mp for NLNp, and Cpq for NKpNq.

2 Basic lemmas

2.1 (Weakening) If $X \models Y$, then $X, W \models Y, V$.

Proof: Suppose $X \models Y$. Then $\bigcap_{p \in Y} \mathrm{I}(p) \subseteq \mathrm{I}(X)$. Since $\mathrm{I}(X) \subseteq \mathrm{I}(X, W)$ and $\bigcap_{p \in Y \cup V} \mathrm{I}(p) \subseteq \bigcap_{p \in Y} \mathrm{I}(p)$, we know $\bigcap_{p \in Y \cup V} \mathrm{I}(p) \subseteq \mathrm{I}(X, W)$. Thus $X, W \models Y, V$.

2.2 (Cut) If $X \models q, Y$ and $q, W \models V$ then $X, W \models Y, V$.

Proof: We want $\bigcap_{p \in Y \cup V} \mathrm{I}(p) \subseteq \mathrm{I}(X, W)$. So suppose $S \in \bigcap_{p \in Y \cup V} \mathrm{I}(p)$. We then want $S \in \mathrm{I}(X, W)$. This is equivalent to $S \cup W \in \mathrm{I}(X)$, which—because $X \models q, Y$—holds if both $S \cup W \in \mathrm{I}(q)$ and $S \cup W \in \mathrm{I}(p)$ for all $p \in Y$.

Now, because $q, W \models V$ and by supposition $S \in \mathrm{I}(p)$ for all $p \in V$, we know $S \in \mathrm{I}(W, q)$ and thus $S \cup W \in \mathrm{I}(q)$. We also know by supposition that $S \in \mathrm{I}(p)$ for all $p \in Y$ and so $S \cup W \in \mathrm{I}(p)$ for all $p \in Y$.

3 Some modal properties

We begin with two small points:

3.1 $\exists Y[Y \not\models \emptyset \ \& \ Y \not\models p]$ iff $\not\models p$.

Proof: ($\Rightarrow$) Instantiate to get $X \not\models p$. This implies $\not\models p$.

($\Leftarrow$) We show the contrapositive. Suppose $\forall Y(Y \models \emptyset$ or $Y \models p)$. Since $Y \models \emptyset$ implies $Y \models p$ we have $\forall Y(Y \models p)$. Then if X is incompatible with p it is incompatible with everything and so $\models p$.

3.2 $\mathrm{L}p \models \emptyset$ iff $\models \emptyset$ or $\not\models p$.

Proof: Instantiating the L Introduction rule with $\emptyset$ as X we get $\mathrm{L}p \models \emptyset$ iff $\models \emptyset$ or $\exists Y[Y \not\models \emptyset \ \& \ Y \not\models p]$. By 3.1 the latter disjunct is equivalent to $\not\models p$, and we thus have $\mathrm{L}p \models \emptyset$ iff $\models \emptyset$ or $\not\models p$.

Note that the disjunct $\models \emptyset$ ("the True implies the False") is not necessarily false. It is equivalent to $P(L) \subseteq \mathrm{Inc}$, which condition is fulfilled in the degenerate case in which every set of sentences is incoherent.

We now get the **basic observation** about modal formulae:

3.3 $X, \mathrm{L}p \models \emptyset \Leftrightarrow X \models \emptyset$ or $\mathrm{L}p \models \emptyset$.

Proof: $(\Rightarrow)$ $X, \mathrm{L}p \models \emptyset$ is by definition equivalent to $X \models \emptyset$ or $\exists Y[X, Y \not\models \emptyset$ and $Y \not\models p]$. Since $X, Y \not\models \emptyset$ implies $Y \not\models \emptyset$ we know $\exists Y[X, Y \not\models \emptyset \;\&\; Y \not\models p]$ implies $\exists Y[Y \not\models \emptyset$ and $Y \not\models p]$, which, instantiating L Introduction with $X = \emptyset$, implies $\mathrm{L}p \models \emptyset$.

$(\Leftarrow)$ This follows from Persistence.

It is now easy to prove:

3.4 (Necessitation) $\models p \Rightarrow \models \mathrm{L}p$.

Proof: Suppose $\models p$ and $X, \mathrm{L}p \models \emptyset$. We want X incoherent. By the basic observation either $X \models \emptyset$, as desired, or $\mathrm{L}p \models \emptyset$, in which case either $\models \emptyset$ or p. In the former case every set is incoherent, including X. The latter case contradicts our supposition and so can't occur.

The entailment that corresponds to the T axiom $\mathrm{CL}pp$ is also easy:

3.5 $\mathrm{L}p \models p$.

Proof: Suppose not. Then there is some Z such that $Z, p \models \emptyset$ and $Z, \mathrm{L}p \not\models \emptyset$. From the latter it follows that $Z \not\models \emptyset$ and $\mathrm{L}p \not\models \emptyset$. From $\mathrm{L}p \not\models \emptyset$ it follows that $\models p$ by 3.2. But then since $Z, p \models \emptyset$ and p is valid, we know that $Z \models \emptyset$, which implies $Z, \mathrm{L}p \models \emptyset$, which is a contradiction.

Using 3.5 and Cut we easily get the following useful rule:

3.6 $\models \mathrm{L}p \Rightarrow \models p$.

We can also extend the basic observation:

3.7 $X, \mathrm{L}p \models Y$ iff $X \models Y$ or $\mathrm{L}p \models \emptyset$.

Proof: $X, \mathrm{L}p \models Y$ is $\forall Z(Z \in \bigcap_{p \in Y} \mathrm{I}(p) \Rightarrow X, \mathrm{L}p, Z \models \emptyset)$, which by 3.3 is $\forall Z(Z \in \bigcap_{p \in Y} \mathrm{I}(p) \Rightarrow X, Z \models \emptyset$ or $\mathrm{L}p \models \emptyset)$. This in turn is equivalent to $\forall Z(Z \in \bigcap_{p \in Y} \mathrm{I}(p) \Rightarrow X, Z \models \emptyset)$ or $\mathrm{L}p \models \emptyset$, which is $X \models Y$ or $\mathrm{L}p \models \emptyset$.

We can also show that with respect to any particular incompatibility frame, every necessary proposition is either contradictory or valid:

3.8 $\mathrm{L}p \models \emptyset$ or $\models \mathrm{L}p$.

Proof: Suppose $\mathrm{L}p \not\models \emptyset$. Then $\models p$ by 3.2 and so $\models \mathrm{L}p$ by 3.4.

Of course, this does not mean that every necessary proposition is either incoherent in all frames or valid in all frames, but only that it is incoherent-or-valid in all frames.

The next result is a dual of 3.7:

3.9 $X \models Y, \mathrm{L}p$ iff $X \models Y$ or $\models \mathrm{L}p$.

Proof: ($\Rightarrow$) Suppose not. Then $X \models Y$ and $\not\models \mathrm{L}p$. By 3.8 we then know $\mathrm{L}p \models \emptyset$. Since $X \not\models Y$ there is some $Z \in \bigcap_{p \in Y} \mathrm{I}(p)$ with $Z \notin \mathrm{I}(X)$. Since $\mathrm{L}p \models \emptyset$ we know Z is incompatible with $\mathrm{L}p$. Since $X \models Y, \mathrm{L}p$ we then have Z incompatible with X, which is a contradiction.

($\Leftarrow$) This follows from Weakening.

4 Semantic reduction

We can, for any given entailment $X \models Y$, show that it is equivalent either to another entailment that mentions fewer connectives or to a Boolean combination of such entailments. We call these equivalences "reduction schemata." They greatly facilitate theorem proving, and their existence makes possible metatheoretical results such as the semantic reduction material in the next section.

4.1 Reduction schemata for non-modal connectives

4.1.1 (Left Negation) $X \models Y, p$ iff $X, \mathrm{N}p \models Y$.

Proof: ($\Rightarrow$) Suppose $X \models Y, p$. We want $\bigcap_{r \in Y} \mathrm{I}(r) \subseteq \mathrm{I}(X, \mathrm{N}p)$. Now suppose $Z \in \bigcap_{r \in Y} \mathrm{I}(r)$. Then $Z \cup \{\mathrm{N}p\} \in \bigcap_{r \in Y} \mathrm{I}(r)$. We also know $p \in \mathrm{I}(\mathrm{N}p)$ and so $Z \cup \{\mathrm{N}p\} \in \mathrm{I}(p)$. Then since $X \models p, Y$ we have $Z \cup \{\mathrm{N}p\} \in \mathrm{I}(X)$ and so $Z \in \mathrm{I}(X, \mathrm{N}p)$.

($\Leftarrow$) Suppose $X, \mathrm{N}p \models Y$. We want $\bigcap_{r \in Y \cup \{p\}} \mathrm{I}(r) \subseteq \mathrm{I}(X)$. So suppose $Z \in \bigcap_{r \in Y \cup \{p\}} \mathrm{I}(r)$. Since $X, \mathrm{N}p \models Y$ we have $Z \in \mathrm{I}(X, \mathrm{N}p)$. Then $Z, X \models p$. Since $Z \in \mathrm{I}(p)$ Cut gives $Z \in \mathrm{I}(Z, X)$ and thus $Z \in \mathrm{I}(X)$.

4.1.2 (Right Negation) $X \models Y, \mathrm{N}p$ iff $X, p \models Y$.

Proof: (⇒) Suppose $X \models Y, \mathrm{N}p$. We want $\bigcap_{r \in Y} \mathrm{I}(r) \subseteq \mathrm{I}(X, p)$. So suppose $Z \in \bigcap_{r \in Y} \mathrm{I}(r)$. Then $Z \cup \{p\} \in \bigcap_{r \in Y} \mathrm{I}(r)$. We also know $Z \cup \{p\} \in \mathrm{I}(\mathrm{N}p)$. Since $X \models Y, \mathrm{N}p$ it follows that $Z \cup \{p\} \in \mathrm{I}(X)$, or $Z \in \mathrm{I}(X, p)$.

(⇐) Suppose $X, p \models Y$. We want $\bigcap_{r \in Y \cup \{\mathrm{N}p\}} \mathrm{I}(r) \subseteq \mathrm{I}(X)$. So suppose $Z \in \bigcap_{r \in Y \cup \{\mathrm{N}p\}} \mathrm{I}(r)$. Then since $Z \in \bigcap_{r \in Y} \mathrm{I}(r)$ and $X, p \models Y$ we have $Z \in \mathrm{I}(X, p)$ or $Z, X, p \models \emptyset$. Since $Z \in \mathrm{I}(\mathrm{N}p)$ we have $Z \models p$. From $Z, X, p \models \emptyset$ and $Z \models p$ we can apply Cut to get $Z, X \models \emptyset$, or $Z \in \mathrm{I}(X)$, as desired.

4.1.3 (Left Conjunction) $X, \mathrm{K}pq \models Y$ iff $X, p, q \models Y$.

Proof: By definition $\mathrm{I}(X, \mathrm{K}pq) = \mathrm{I}(X, p, q)$. The result follows immediately.

4.1.4 (Right Conjunction) $X \models Y, \mathrm{K}pq$ iff $X \models Y, p$ and $X \models Y, q$.

Proof: (⇒) Suppose $X \models Y, \mathrm{K}pq$. Then $\bigcap_{r \in Y \cup \{\mathrm{K}pq\}} \mathrm{I}(r) \subseteq \mathrm{I}(X)$. Equivalently, if $Z \in \mathrm{I}(p, q)$ and $Z \in \bigcap_{r \in Y} \mathrm{I}(r)$ then $Z \in \mathrm{I}(X)$. But $\mathrm{I}(p) \subseteq \mathrm{I}(p, q)$. Then if $Z \in \mathrm{I}(p)$ and $Z \in \bigcap_{r \in Y} \mathrm{I}(r)$ it follows that $Z \in \mathrm{I}(p, q)$ and $Z \in \bigcap_{r \in Y} \mathrm{I}(r)$ and so $Z \in \mathrm{I}(X)$. Thus $X \models Y, p$. We can argue similarly to get $X \models Y, q$.

(⇐) Suppose $X \models Y, p$ and $X \models Y, q$. We want $\bigcap_{r \in Y \cup \{\mathrm{K}pq\}} \mathrm{I}(r) \subseteq \mathrm{I}(X)$. So suppose $Z \in \mathrm{I}(\mathrm{K}pq)$ and $Z \in \bigcap_{r \in Y} \mathrm{I}(r)$. If $Z \in \mathrm{I}(\mathrm{K}pq)$ then $Z, p, q \models \emptyset$. By Cut and the fact that $X \models Y, p$ we then have $Z, X, q \models Y$. Applying Cut again, this time with $X \models Y, q$, we get $Z, X \models Y$. Since by supposition $Z \in \bigcap_{r \in Y} \mathrm{I}(r)$ we then know $Z \in \mathrm{I}(Z, X)$, which is $Z \in \mathrm{I}(X)$, as desired.

4.2 Reduction schemata for modal connectives

4.2.1 (Left Necessity) $X, \mathrm{L}p \models Y$ iff $X \models Y$ or $\not\models p$.

Proof: If we are in the degenerate frame everything implies everything else, so the result holds. If not, it follows from 3.7 and 3.2.

4.2.2 (Right Necessity) $X \models Y, \mathrm{L}p$ iff $X \models Y$ or $\models p$.

Proof: Apply 3.9, then 3.4 and 3.6.

5 *Incompatibilities for extensions of a language*

5.1 Motivation Crucial to the compositionality of meaning is that the semantic values of logically complex sentences be reducible to the semantic values of their constituents. In the framework of incompatibility logic, however, meaning is holistic, and so this reduction cannot proceed sentence by sentence. What we want instead is to be show how the frame for a language with logically complex sentences can be reduced to the frame for a syntactically less complex fragment of the language.

Suppose $L \subseteq L'$. Say that L' is a *proper extension* of L if L' is a proper language in the sense of section 1 and all atomic formulae in L' are contained in L (thus every formula in L' is built from formulae in L and all the intermediate formulae in the construction are also in L'). Our problem then, can be formulated thus: Given a frame Inc for a language L, and given a language L' properly extending L, what is the frame Inc$'$ for L' that is determined by Inc?

One property that Inc$'$ should have is that it should agree with Inc about the semantic properties of vocabulary in L. For example, if L is an atomic language and L' contains in addition logically complex formulae formed from the sentences of L, Inc$'$ should agree with Inc about which entailments $X \models Y$ hold for those X, Y that are sets of atomic formulae. This agreement is a form of *inferential conservativeness*:

Let $L \subseteq L'$ and let Inc be a frame for L. Then a frame Inc$'$ for L' is *inferentially conservative (for short, IC) with respect to Inc* if, for $X, Y \subseteq L$, $X \models_{\text{Inc}} Y \Leftrightarrow X \models_{\text{Inc}'} Y$.

For finite languages, the requirement of inferential conservativeness is sufficient to determine frames for properly extending languages. That is, given a finite L, a frame Inc for L, and an extending language L', it is sufficient to determine Inc$'$ simply to require that Inc$'$ be inferentially conservative with respect to Inc.

If L is infinite, however, it ceases to be the case that atomic frames generate unique inferentially conservative extensions. (Inferential conservativeness suffices for infinite languages as well if we stipulate that all incoherent sets be finite. But there is no compelling a priori reason for such a stipulation.) Given a frame Inc for some language L, and a language L' extending L, there can be multiple frames for L' that conservatively extend Inc. But as algebraic experience might lead us to expect, there is always a single *smallest* frame for L' that is inferentially conservative with respect to Inc (where smallest has the sense of, contained in every other frame for L' that is IC with respect to Inc). There might, depending on the case, be *other* frames for L' that are inferentially conservative with respect to Inc, but each of these other frames properly contains Inc$'$, i.e., they can be obtained from Inc$'$ by *adding* further semantic information to Inc$'$ in the form of stipulating that additional sets of formulae are incoherent in addition to those deemed incoherent by Inc$'$. Accordingly, we view Inc$'$, but not these other frames, as determined solely by the semantic information contained in Inc.

Thus, we have the following definition that encapsulates all that we need for a sensible theory of semantic reduction:

Let L' be a proper extension of L and let Inc be a frame for L. The *frame for L' determined by Inc* is the smallest frame for L' that is IC with respect to Inc.

5.2 Existence of the determined frame We now show that the determined frame exists. (If it does exist, it is immediate from the definition that it is unique.) The following results can be heuristically summarized as follows. Suppose we are given a language L, a frame Inc for L, and a proper extension L' of L. Now suppose a frame for L' contains only subsets of L' whose incoherence can be shown to follow from Inc by a finite number of applications of the reduction schemata of section 4. Such a frame is contained in every frame for L' that is IC with respect to Inc, and hence it is the unique smallest frame for L' that is IC with respect to Inc (5.2.1–5.2.2). Further, such a frame for L' always exists (5.2.3–5.2.7). Thus the frame for L' determined by Inc exists (5.2.8).

Call a frame Inc *finitary over* L if for each $X \in$ Inc, $X - L$ is finite (that is, X contains only finitely many formulae not in L). Call a frame Inc *genetically finitary over* L if for each $X \in$ Inc there is some $X' \subseteq X$ such that $X' \in$ Inc and $X' - L$ is finite.

5.2.1 (Semantic Reduction Lemma) Given a language L and frame Inc, let $L_\emptyset$ be a fragment of L such that L is a proper extension of $L_\emptyset$. Let $X, Y \subseteq L$ be such that $X - L_\emptyset$ and $Y - L_\emptyset$ are finite. Then there is a Boolean function F on n propositions and sets of sentences $X_i, Y_i \subseteq L_\emptyset$ such that $X \models_{\mathrm{Inc}} Y$ iff $F(X_1 \models_{\mathrm{Inc}} Y_1;$

$\ldots; X_n \models_{\text{Inc}} Y_n)$. Further, $F(X_1 \models_{\text{Inc}} Y_1; \ldots; X_n \models_{\text{Inc}} Y_n)$ can be chosen on the basis of just the syntax of the members of X and Y.

Proof: We actually show a slightly stronger result, viz. $F(X_1 \models_{\text{Inc}} Y_1; \ldots; X_n \models_{\text{Inc}} Y_n)$ iff $G(X'_1 \models_{\text{Inc}} Y'_1; \ldots; X'_m \models_{\text{Inc}} Y'_m)$, where $X_i, Y_i \subseteq L$; $X'_i, Y'_i \subseteq L_\emptyset$; and $X_i - L_\emptyset$ and $Y_i - L_\emptyset$ are finite for every i.

Fix an ordering of the formulae of L. We prove the result by induction on the number of connectives contained in all the X_i, Y_i.

Given $F(X_1 \models_{\text{Inc}} Y_1; \ldots; X_n \models_{\text{Inc}} Y_n)$ choose the first $X_i \models_{\text{Inc}} Y_i$ that mentions a formula not in $L_\emptyset$ and choose the first such formula in $X_i \models_{\text{Inc}} Y_i$ according to our chosen ordering. We have six possibilities, according as the major operator is N, K, or L, and according as the formula is in the ~~consequent~~ or the ~~antecedent~~ of the entailment. In the first case (an N-formula in antecedent position), we have

antecedent

consequent

$$F(X_1 \models_{\text{Inc}} Y_1; \ldots; X_n \models_{\text{Inc}} Y_n) \text{ iff}$$
$$F(X_1 \models_{\text{Inc}} Y_1; \ldots; \text{N}p, Z_i \models_{\text{Inc}} Y_i; \ldots; X_n \models_{\text{Inc}} Y_n),$$

where $X_i = \{\text{N}p\} \cup Z_i$. Applying 4.1.2 we get:

$$F(X_1 \models_{\text{Inc}} Y_1; \ldots; X_n \models_{\text{Inc}} Y_n) \text{ iff}$$
$$F(X_1 \models_{\text{Inc}} Y_1; \ldots; Z_i \models_{\text{Inc}} Y_i, p; \ldots; X_n \models_{\text{Inc}} Y_n).$$

The right-hand side is a Boolean combination of entailments that mentions one fewer connective than the left-hand side, so we may apply the induction hypothesis to get:

$$F(X_1 \models_{\text{Inc}} Y_1; \ldots; X_n \models_{\text{Inc}} Y_n) \text{ iff } G(X'_1 \models_{\text{Inc}} Y'_1; \ldots; X'_m \models_{\text{Inc}} Y'_m)$$

as desired. The remaining five cases are treated similarly, applying 4.1.1, 4.1.3, 4.1.4., 4.2.1, or 4.2.2 as the case may be.

5.2.2 Let Inc be a frame for L and let L' properly extend L. Let Inc$'$ be a frame for L' that is genetically finitary over L and IC with respect to Inc. If Inc$''$ is another frame for L' that is IC with respect to Inc, then Inc$' \subseteq$ Inc$''$.

Proof: Suppose $X \in$ Inc$'$. Since Inc$'$ is genetically finitary over L, there is some $X' \subseteq X$ such that $X' \in$ Inc$'$ and $X' - L$ is finite. Then by 5.2.1 there exists some F and some $X_i, Y_i \subseteq L$ such that $X' \models_{\text{Inc}'} \emptyset$ implies $F(X_1 \models_{\text{Inc}'} Y_1; \ldots; X_n \models_{\text{Inc}'} Y_n)$. Since Inc$'$ is IC with respect to Inc, this implies $F(X_1 \models_{\text{Inc}} Y_1; \ldots; X_n \models_{\text{Inc}} Y_n)$. Since Inc$''$ is IC with respect to Inc, we have $F(X_1 \models_{\text{Inc}''} Y_1; \ldots; X_n \models_{\text{Inc}''} Y_n)$. By 5.2.1 again $X' \models_{\text{Inc}''} \emptyset$, or $X' \in$ Inc$''$. By Persistence, $X \in$ Inc$''$. Thus Inc$' \subseteq$ Inc$''$.

5.2.3 Let Inc be a frame for L. Let $L' = L \cup \{\text{K}pq\}$ for some $p, q \in L$. Define $F(X)$ as follows. If $X = X' \cup \{\text{K}pq\}$ then $F(X) = X' \cup \{p, q\}$; otherwise $F(X) = X$. Let $X \in$ Inc$'$ if $F(X) \in$ Inc. Then (i) Inc$'$ is a frame for L'; (ii) Inc$'$ is IC with respect to Inc; and (iii) Inc$'$ is finitary over L.

Proof: (i) We verify the frame axioms.

(Persistence) Suppose $X \in$ Inc$'$ and $X \subseteq Y$. Then $F(X) \in$ Inc and so by Persistence for Inc, $F(Y) \in$ Inc. Then $Y \in$ Inc$'$.

(NI) Suppose $X \cup \{\text{N}r\} \in$ Inc$'$. We want $X \models_{\text{Inc}'} r$. Suppose then that $Z \cup \{r\} \in$ Inc$'$. It now suffices to show that $Z \cup X \in$ Inc$'$. Since $X \cup \{\text{N}r\} \in$ Inc$'$ we know $F(X) \cup \{\text{N}r\} \in$ Inc. Then $F(X) \models_{\text{Inc}} r$. Since $Z \cup \{r\} \in$ Inc$'$ we know $F(Z) \cup \{r\} \in$ Inc. Then $F(X) \cup F(Z) = F(X \cup Z) \in$ Inc, and so $X \cup Z \in$ Inc$'$.

Suppose $X \models_{\text{Inc}'} r$. We want $X \cup \{\text{N}r\} \in$ Inc$'$. Since N$r \neq$ Kqp, N$r \in L$. Then $\{r, \text{N}r\} \in$ Inc, so $\{r, \text{N}r\} \in$ Inc$'$. Since $X \models_{\text{Inc}'} r$, it follows that $X \cup \{\text{N}r\} \in$ Inc$'$.

(KI) $X \cup \text{K}rs \in$ Inc$'$ iff $F(X \cup \text{K}rs) \in$ Inc
iff $F(X) \cup \{\text{r,s}\} \in$ Inc
iff $X \cup \{r, s\} \in$ Inc$'$

(LI) Suppose $X \cup \{\text{L}r\} \in$ Inc$'$. We want $X \in$ Inc$'$ or $\exists Y[X \cup Y \notin$ Inc$'$ & $Y \not\models_{\text{Inc}'} r]$.

Suppose $X \notin$ Inc$'$. We show $\exists Y[X \cup Y \notin$ Inc$'$ & $Y \not\models_{\text{Inc}'} r]$. Since $X \cup \{\text{L}r\} \in$ Inc$'$ we know $F(X) \cup \{\text{L}r\} \in$ Inc. Since $X \notin$ Inc$'$ we know $F(X) \notin$ Inc. Then $\exists Y[F(X) \cup Y \notin$ Inc & $Y \not\models_{\text{Inc}} r]$.

Instantiate to some W, so that $F(X) \cup W \notin \text{Inc}$ and $W \not\models_{\text{Inc}} r$. Since K$pq \notin L$ we know $W = F(W)$, so $F(X \cup W) \notin \text{Inc}$ and $F(W) \not\models_{\text{Inc}} r$. Then there is some Z such that $Z \cup \{r\} \in \text{Inc}$ and $Z \cup F(W) \notin \text{Inc}$. Note that K$pq \notin Z$ because K$pq \notin L$ and $Z \subseteq L$. Then $Z \cup \{r\} \in \text{Inc}'$ and $Z \cup W \notin \text{Inc}'$. Thus $W \not\models_{\text{Inc}'} r$. Thus $\exists Y[X \cup Y \notin \text{Inc}' \;\&\; Y \not\models_{\text{Inc}'} r]$.

Now suppose $X \in \text{Inc}'$ or $\exists Y[X \cup Y \notin \text{Inc}' \;\&\; Y \not\models_{\text{Inc}'} r]$. We want $X \cup \{\text{L}p\} \in \text{Inc}'$. If $X \in \text{Inc}'$ then the result follows by Persistence. So suppose $\exists Y[X \cup Y \notin \text{Inc}' \;\&\; Y \not\models_{\text{Inc}'} r]$. Instantiate to some W to get $X \cup W \notin \text{Inc}'$ and $W \not\models_{\text{Inc}'} r$. Then $F(X \cup W) = F(X) \cup F(W) \notin \text{Inc}'$. Now since $W \not\models_{\text{Inc}'} r$ there is some Z such that $Z \cup \{r\} \in \text{Inc}'$ but $Z \cup W \notin \text{Inc}'$. Then $F(Z) \cup \{r\} \in \text{Inc}$ and $F(Z \cup W) = F(Z) \cup F(W) \notin \text{Inc}$. Thus $F(W) \not\models_{\text{Inc}} r$. It follows that $\exists Y[F(X) \cup Y \notin \text{Inc} \;\&\; Y \not\models_{\text{Inc}} r]$ and so $F(X) \cup \{\text{L}p\} = F(X \cup \{\text{L}p\}) \in \text{Inc}$. Thus $X \cup \{\text{L}p\} \in \text{Inc}'$, as desired.

(ii) Suppose $X, Y \subseteq L$. We want $X \models_{\text{Inc}'} Y \Leftrightarrow X \models_{\text{Inc}} Y$.

($\Rightarrow$) Suppose $X \models_{\text{Inc}'} Y$ and $Z \cup \{y_i\} \in \text{Inc}$ for each $y_i \in Y$. We want $Z \cup X \in \text{Inc}$. Since $Y, Z \subseteq L$ we know $F(Z \cup \{y_i\}) = Z \cup \{y_i\}$. Then $F(Z \cup \{y_i\}) \in \text{Inc}$ and so $Z \cup \{y_i\} \in \text{Inc}'$. Since $X \models_{\text{Inc}'} Y$, $Z \cup X \in \text{Inc}'$. By construction, $F(Z \cup X) = Z \cup X \in \text{Inc}$.

($\Leftarrow$) Suppose $X \models_{\text{Inc}} Y$ and $Z \cup \{y_i\} \in \text{Inc}'$ for each $y_i \in Y$. Then $F(Z \cup \{y_i\}) \in \text{Inc}$, and since $Y \subseteq L$, $F(Z) \cup \{y_i\} \in \text{Inc}$. Since $X \models_{\text{Inc}} Y$, $F(Z) \cup X = F(Z \cup X) \in \text{Inc}$. But then $Z \cup X \in \text{Inc}'$.

(iii) Consider some $X \in \text{Inc}'$. The only formula not in L that X can contain is Kpq, so $X - L$ is finite.

5.2.4 Let Inc be a frame for L. Let $L' = L \cup \{\text{N}p\}$ for some $p \in L$. Then let $X \in \text{Inc}'$ if $X \in \text{Inc}$ and let $X \cup \{\text{N}p\} \in \text{Inc}$ if $X \models_{\text{Inc}} p$. Then (i) Inc′ is a frame for L′; (ii) Inc is IC with respect to Inc; and (iii) Inc′ is finitary over L.

Proof: As in 5.2.3.

5.2.5 Let Inc be a frame for L. Let $L' = L \cup \{\text{L}p\}$ for some $p \in L$. Then let $X \in \text{Inc}'$ if $X \in \text{Inc}$ and let $X \cup \{\text{L}p\}$ if $\not\models_{\text{Inc}} p$. Then (i) Inc′ is a frame for L'; (ii) Inc is IC with respect to Inc; and (iii) Inc′ is finitary over L.

Proof: As in 5.2.3.

5.2.6 Let Inc be a frame for L. Let $< p_1, p_2, \ldots >$ be a sequence of formulae such that each $L_i = L \cup \{p_1, \ldots, p_i\}$ is a proper extension of L; let $L_0 = L$. Then (i) there exists a frame Inc_i for each L_i; (ii) Inc_i is IC with respect to Inc_j for each $j \leq i$; and (iii) Inc_i is finitary over L_j for each $j \leq i$.

Proof: By induction on the L_i.

(Base case) L_0 satisfies (i)–(iii) immediately.

(Inductive case) Assume the claim is true for $L_0, \ldots, L_{n-1}$, and that we thus have for each $i \leq n-1$ a frame Inc_i for L_i satisfying (ii) and (iii). Now, p_n is either Kpq, Np, or Lp, for some p (and possibly q) in L_{n-1}. Then by 5.2.3, 5.2.4, or 5.2.5, as the case may be, we have a frame Inc_n for L_n. Hence (i) is satisfied. We also know that Inc_n is IC with respect to Inc_{n-1} and that it is finitary over L_{n-1}. We now show that Inc_n satisfies (ii) and (iii).

(ii) Consider some $X, Y \subseteq L_j$ for $j \leq n$. Since $L_j \subseteq L_{n-1}$, we have $X, Y \subseteq L_{n-1}$. And since Inc_n is IC with respect to Inc_{n-1}, we have $X \models_{\text{Inc}_n} Y$ iff $X \models_{\text{Inc}_{n-1}} Y$. But Inc_{n-1} is IC with respect to Inc_j, so $X \models_{\text{Inc}_{n-1}} Y$ iff $X \models_{\text{Inc}_j} Y$. Thus $X \models_{\text{Inc}_n} Y$ iff $X \models_{\text{Inc}_j} Y$, which is to say that Inc_n is IC with respect to Inc_j.

(iii) Consider some $X \in \text{Inc}_n$. For $i \leq n$ the formulae in X that are not in L_i are a subset of $L_n - L_i$, which is finite. Thus X is finitary over L_i.

5.2.7 Let Inc be a frame for L. Let $< p_1, p_2, \ldots >$ be a sequence of formulae such that each $L_i = L \cup \{p_1, \ldots, p_i\}$ is a proper extension of L. Let $L' = L \cup \{p_1, p_2, \ldots\}$. Then (i) there exists a frame Inc′ for L'; (ii) Inc′ is IC with respect to Inc; and (iii) Inc′ is genetically finitary over L.

Proof: Let $L_0 = L$. By 5.2.6, there is a frame Inc_i for each L_i; Inc_i is IC with respect to Inc_j for all $j \leq i$; and Inc_i is finitary over L_j. Let $X \in \text{Inc}'$ if there is some $X' \subseteq X$ such that $X' \in \text{Inc}_i$ for some i. We claim that Inc′ is the desired frame.

Lemma: Let $X \cup \{p\} \in \text{Inc}'$. Then there is some $X' \subseteq X$ and some i such that $X' \cup \{p\} \in \text{Inc}_i$.

Proof: If $X \cup \{p\} \in \text{Inc}'$ then there is some $W \subseteq X \cup \{p\}$ and some j such that $W \in \text{Inc}_j$. Then there is some k, with $j \leq k$, such that $p \in L_k$. Since Inc_k is IC with respect to Inc_j, $W \in \text{Inc}_k$ and so $W \cup \{p\} \in \text{Inc}_k$ by Persistence. Take $X' = W - \{p\}$. QED (lemma).

In the following we use the lemma without notice.

(i) We verify the frame axioms.

(Persistence) Suppose $X \in \text{Inc}'$ and $X \subseteq Y$. Then there is some $X' \subseteq X$ such that $X \in \text{Inc}_i$ for some i. But $X' \subseteq Y$ and so $Y \in \text{Inc}'$ by construction.

(NI) Suppose $X \cup \{\text{N}p\} \in \text{Inc}'$. We want $X \models_{\text{Inc}'} p$. Suppose $Z \cup \{p\} \in \text{Inc}'$. It suffices then to show that $Z \cup X \in \text{Inc}'$. Since $X \cup \{\text{N}p\} \in \text{Inc}'$ there is some $X' \subseteq X$ and some j such that $X' \cup \{\text{N}p\} \in \text{Inc}_j$. Similarly, there is some k and some $Z' \subseteq Z$ such that $Z' \cup \{p\} \in \text{Inc}_k$. Let $l = \max\{k, j\}$. Then $X' \cup \{\text{N}p\} \in \text{Inc}_l$ and $Z' \cup \{p\} \in \text{Inc}_l$. Since $X' \cup \{\text{N}p\} \in \text{Inc}_l$ we know $X' \models_{\text{Inc}_l} p$. Then $X' \cup Z' \in \text{Inc}_l$ and so $X \cup Z \in \text{Inc}'$.

Suppose $X \models_{\text{Inc}'} p$. We want $X \cup \{\text{N}p\} \in \text{Inc}'$. Choose some i such that $\text{N}p \in L_i$. Then $\{p, \text{N}p\} \in \text{Inc}_i$ and so $\{p, \text{N}p\} \in \text{Inc}'$. Then $X \cup \{\text{N}p\} \in \text{Inc}'$.

(KI) Suppose $X \cup \{\text{K}pq\} \in \text{Inc}'$. We want $X \cup \{p, q\} \in \text{Inc}'$. Since $X \cup \{\text{K}pq\} \in \text{Inc}'$ there is some $X' \subseteq X$ and some i such that $X' \cup \{\text{K}pq\} \in \text{Inc}_i$. Then $X' \cup \{p, q\} \in \text{Inc}_i$ and so $X \cup \{p, q\} \in \text{Inc}'$.

The converse is proved similarly.

(LI) Suppose $X \cup \{\text{L}p\} \in \text{Inc}'$. We want $X \in \text{Inc}'$ or $\exists Y[X \cup Y \notin \text{Inc}' \,\&\, Y \not\models_{\text{Inc}'} p]$.

Suppose $X \notin \text{Inc}'$. We show $\exists Y[X \cup Y \notin \text{Inc}' \,\&\, Y \not\models_{\text{Inc}'} p]$. Since $X \cup \{\text{L}p\} \in \text{Inc}'$ there is some $X' \subseteq X$ and some j such that $X' \cup \{\text{L}p\} \in \text{Inc}_j$. Then $X' \in \text{Inc}_j$ or $\not\models_{\text{Inc}_j} p$. Since $X \notin \text{Inc}'$, we know $X' \notin \text{Inc}_j$; thus $\not\models_{\text{Inc}_j} p$. Then there is some Z such that $Z \cup \{p\} \in \text{Inc}_j$ but $Z \notin \text{Inc}_j$. Thus $Z \cup \{p\} \in \text{Inc}'$. Further, we claim $Z \notin \text{Inc}'$. For suppose $Z \in \text{Inc}'$. Then there is some $Z' \subseteq Z$ and some k such that $Z' \in \text{Inc}_k$. Let $l = \max\{j, k\}$. Then $Z' \in \text{Inc}_l$ because Inc_l is IC with respect to Inc_k. But since $L_j \subseteq L_l$ we know $Z \subseteq L_l$, so $Z \in \text{Inc}_l$ by Persistence. But since $Z \subseteq L_j$ and Inc_l is IC with respect to Inc_j, we have $Z \in \text{Inc}_j$, which is a contradiction. Thus $X \notin \text{Inc}'$ and $\not\models_{\text{Inc}'} p$. Generalizing with $Y = \emptyset$ we have $\exists Y[X \cup Y \notin \text{Inc}' \;\&\; Y \not\models_{\text{Inc}'} p]$, as desired.

Now suppose $X \in \text{Inc}'$ or $\exists Y[X \cup Y \notin \text{Inc}' \;\&\; Y \not\models_{\text{Inc}'} p]$. We want $X \cup \{\text{L}p\} \in \text{Inc}'$.

If $X \in \text{Inc}'$ the result follows by Persistence. So suppose $X \notin \text{Inc}'$ and $\exists Y[X \cup Y \notin \text{Inc}' \;\&\; Y \not\models_{\text{Inc}'} p]$. Instantiate to some W, to give $X \cup W \notin \text{Inc}'$ and $W \not\models_{\text{Inc}'} p$. Then there is some Z such that $Z \cup \{p\} \in \text{Inc}'$ and $Z \cup W \notin \text{Inc}'$. Since $Z \cup W \notin \text{Inc}'$ we know $Z \notin \text{Inc}'$. Since $Z \cup \{p\} \in \text{Inc}'$ there is some $Z' \subseteq Z$ and some i such that $Z' \cup \{p\} \in \text{Inc}_i$. Also, since $Z \notin \text{Inc}'$ we know $Z' \notin \text{Inc}_i$. Then $\not\models_{\text{Inc}_i} p$. Pick some subset $X' \subseteq X$ such that $X' \subseteq L_i$. Since $X \notin \text{Inc}'$ we know $X' \notin \text{Inc}_i$. Then we have $X' \notin \text{Inc}_i$ and $\not\models_{\text{Inc}_i} p$. Generalizing with $Y = \emptyset$ we have $\exists Y[X' \cup Y \notin \text{Inc}_i \;\&\; Y \not\models_{\text{Inc}_i} p]$. Then $X' \cup \{\text{L}p\} \in \text{Inc}_i$ and so $X \cup \{\text{L}p\} \in \text{Inc}'$ by construction.

(ii) Suppose $X, Y \subseteq L$. We want $X \models_{\text{Inc}'} Y \Leftrightarrow X \models_{\text{Inc}} Y$.

($\Rightarrow$) Suppose $X \models_{\text{Inc}'} Y$ and $Z \cup \{y_i\} \in \text{Inc}$ for each $y_i \in Y$. We want $Z \cup X \in \text{Inc}$. Since $Z \cup \{y_i\} \in \text{Inc} = \text{Inc}_0$ we have $Z \cup \{y_i\} \in \text{Inc}'$ and so $Z \cup X \in \text{Inc}'$. Then there is some $W \subseteq Z \cup X$ and some j such that $W \in \text{Inc}_j$. But Inc_j is IC with respect to Inc and $W \subseteq L$, so $W \in \text{Inc}$. Since $W \subseteq Z \cup X \subseteq L$, it follows by Persistence that $Z \cup X \in \text{Inc}$.

($\Leftarrow$) Suppose $X \models_{\text{Inc}} Y$ and $Z \cup \{y_i\} \in \text{Inc}'$ for each $y_i \in Y$. Then for each y_i there is some $Z'_i \subseteq Z$ and some j such that $Z'_i \cup \{y_i\} \in \text{Inc}_j$. Since there are a finite number of y_i (recall the definition of entailment from 1.1), there is a largest such j; call it k. Since Inc_k is IC with respect to Inc_j for each j, we have $Z'_i \cup \{y_i\} \in \text{Inc}_k$ for all i. Let $Z' = \bigcup_i Z'_i$. We know $Z' \subseteq L_k$ and so by Persistence $Z' \cup \{y_i\} \in \text{Inc}_k$. Since Inc_k is IC with respect to Inc, we know that $X \models_{\text{Inc}_k} Y$. Then $X \cup Z' \in \text{Inc}_k$. Thus $X \cup Z \in \text{Inc}'$, as desired.

(iii) Suppose $X \in \text{Inc}'$. Then there is some $X' \subseteq X$ and some i such that $X' \in \text{Inc}_i$. But Inc_i is finitary over L, so $X' - L$ is finite. In addition $X' \in \text{Inc}'$ by construction.

5.2.8 Let Inc be a frame for L and let L' properly extend L. Then there exists a frame Inc′ that is the frame for L' determined by Inc.

Proof: Let $< p_1, p_2, \ldots >$ be an enumeration of $L' - L$ such that $i \leq j$ implies that p_j is not a subformula of p_i; if $L' - L$ is finite then let the sequence repeat after some n. Since L' is a proper extension of L we know $L \cup \{p_1, \ldots, p_i\}$ is a proper language for every i. Then by 5.2.7 there is a frame Inc′ for L' that is IC with respect to Inc and is genetically finitary over L. By 5.2.2 Inc′ is contained in every frame Inc″ for L' that is IC with respect to Inc. Inc′ is therefore the smallest frame for L' that is IC with respect to Inc, and hence is the frame for L' determined by Inc.

Appendix 2: Logic using the reduction formulae

Appendix 1 showed that the semantic definitions of the principal connectives in terms of incompatibility/incoherence underwrites these six reduction schemata:

LN: $X, \text{N}p \models Y \Leftrightarrow X \models Y, p$.
RN: $X \models Y, \text{N}p \Leftrightarrow X, p \models Y$.

LK: $X, \text{K}pq \models Y \Leftrightarrow X, p, q \models Y$.
RK: $X \models Y, \text{K}pq \Leftrightarrow X \models Y, p$ and $X \models Y, q$.

LL: $X, \text{L}p \models Y \Leftrightarrow X \models Y$ or $\not\models p$.
RL: $X \models Y, \text{L}p \Leftrightarrow X \models Y$ or $\models p$.

Using them, it is easy to show that the incompatibility semantics validates classical logic for N and K (hence for A), and S5 when we add L (and hence M):

1 Negation

1.1 If $\{p\} \in \text{I}(X)$ and $\{\text{N}p\} \in \text{I}(X)$, then $X \in \text{Inc}$.

Proof: If $\{\mathrm{N}p\} \in \mathrm{I}(X)$ then $X \models p$, that is, $\mathrm{I}(p) \subseteq \mathrm{I}(X)$. Since $\{p\} \in \mathrm{I}(X)$, we have $X \in \mathrm{I}(p)$. Thus $X \in \mathrm{I}(X)$, or $X \in \mathrm{Inc}$.

It follows that $\mathrm{N}p \in \mathrm{I}(p)$:

1.2 (Double Negation Equivalence) $\mathrm{NN}p \approx p$.

Proof: $\mathrm{NN}p \models p \Leftrightarrow \emptyset \models \mathrm{N}p, p$ (LN) $\Leftrightarrow p \models p$ (RN).

normal word space

$(\Leftarrow)$ $p \models \mathrm{NN}p \Leftrightarrow p, \mathrm{N}p \models \emptyset$ (RN) $\Leftrightarrow p \models p$ (LN).

1. 1.3 (Contraposition 1) $p \models q \Leftrightarrow \mathrm{N}q \models \mathrm{N}p$.

Proof: $\mathrm{N}q \models \mathrm{N}p \Leftrightarrow \emptyset \models \mathrm{N}p, q$ (LN) $\Leftrightarrow p \models q$ (RN).

1.4 (Material Consistency) $\mathrm{I}(X) \subseteq \mathrm{Inc} \Rightarrow {\sim}\exists Y[Y \in \mathrm{I}(X)\ \&\ \mathrm{I}(Y) \subseteq \mathrm{Inc}]$.

That is, if X and Y are materially incompatible, they cannot both be valid in the sense of having only incompatibles that are self-incompatible. This result, we should note, depends on our not being in the degenerate frame in which all sets are incoherent. In that case, all sets are also valid.

Proof: Suppose we are not in the degenerate frame. By definition, $\mathrm{I}(X) \subseteq \mathrm{Inc} \Leftrightarrow (Z \in \mathrm{I}(X) \rightarrow Z \in \mathrm{I}(Z))$. So only self-incompatible Y could be incompatible with such an X. But since *everything* is incompatible with a self-incompatible Y, it cannot be that $\mathrm{I}(Y) \subseteq \mathrm{Inc}$. For instance, X would be a non-self-incompatible counterexample to $\mathrm{I}(Y) \subseteq Y$. (If X is self-incompatible, it follows that all sets are incoherent, and hence that we are in the degenerate frame.)

1.5 (Formal Consistency) $(X \notin \mathrm{I}(X)$ and $X \models p) \Rightarrow X \not\models \mathrm{N}p$.

Proof: By NI and Partition, $X \in \mathrm{I}(\mathrm{N}p) \Leftrightarrow X \models p$. But then X itself is a counterexample to $X \models \mathrm{N}p$, since $X \in \mathrm{I}(p)$ and $X \notin \mathrm{I}(X)$.

It follows immediately that if $\models p$, then $\not\models \mathrm{N}p$ (assuming, again, that we are not in the degenerate frame). So the incompatibility logic of negation is consistent.

Since incoherent sets entail everything, it also follows that $(X \models p$ and $X \models \mathrm{N}p) \Rightarrow \forall Y[X \models Y]$.

2 Conjunction

2.1 $\mathrm{K}pq \models p$ and $\mathrm{K}pq \models q$.

Proof: By LK, K$pq \models Y \Leftrightarrow p, q \models Y$. So K$pq \models p \Leftrightarrow p, q \models p$, and K$pq \models q \Leftrightarrow p, q \models q$. But $p, q \models p$ and $p, q \models q$ hold by Weakening.

2.2 ($X \models p$ and $X \models q$) $\Leftrightarrow X \models$ Kpq.

Proof: By RK, $X \models Y$, K$pq \Leftrightarrow$ ($X \models Y, p$ and $X \models Y, q$). Letting $Y = \emptyset$, then, $X \models$ K$pq \Leftrightarrow$ ($X \models p$ and $X \models q$).

3 Negation and Conjunction Together

3.1 KpN$p \models Y$.

Proof: Immediate from LK and the final observation under 1.5 above.

Anything that satisfies 1.2, 1.3, 2.1, 2.2, and 3.1 and *distributivity* is classical (Boolean) logic.

3.2 (Distributivity): KpA$qr \approx$ AKpqKpr.

Proof:

(a) A$xy \approx_{\mathrm{df}}$ NKNxNy. So KpA$qr \approx$ AKpqKpr iff KpNKNqNr $\approx$ NKNKpqNKpr.
(b) First direction: KpNKNqN$r \models$ NKNKpqNKpr.
(c) KpNKNqN$r \models$ NKNKpqNKpr iff KpNKNqNr, KNKpqNKpr $\models \emptyset$ (RN).
(d) KpNKNqNr, KNKpqNK$pr \models \emptyset$ iff KpNKNqNr, NKpq, NK$pr \models \emptyset$ (LK) iff p, NKNqNr, NKpq, NK$pr \models \emptyset$ (LK).
(e) p, NKNqNr, NKpq, NK$pr \models \emptyset$ iff p, NKpq, NK$pr \models$ KNqNr (LN) iff p, NK$pr \models$ Kpq, KNqNr (LN).
(f) p, NK$pr \models$ Kpq, KNqNr iff p, NK$pr \models$ Kpq, Nq and p, NK$pr \models$ Kpq, Nr (RK).
(g) p, NK$pr \models$ Kpq, Nq iff p, NK$pr \models$ Nq, p and NK$pr \models$ Nq, q (RK).
(h) p, NK$pr \models$ Kpq, Nr iff p, NK$pr \models$ Nr, p and p, NK$pr \models$ Nr, q (RK).
(i) So, plugging (h) and (g) into (f): p, NK$pr \models$ Kpq, KNqNr iff
    (i) p, NK$pr \models$ Nq, p and
    (ii) p, NK$pr \models$ Nq, q and
    (iii) p, NK$pr \models$ Nr, p and
    (iv) p, NK$pr \models$ Nr, q.

(j) Now (i–i), and (i–iii) hold because $p \models p$. (i–ii) holds because p, NK$pr \models$ Nq, q iff p, NKpr, $q \models q$ (RN), and p, NKpr, $q \models q$ because $q \models q$. (i–iv) holds because p, NK$pr \models$ Nr, q iff $p \models$ Kpr, Nr, q (LN), which, by LK, holds iff $p \models p$, Nr, q and $p \models r$, Nr, q. $p \models p$, Nr, q holds because $p \models p$, and $p \models r$, Nr, q holds since it is equivalent by LN to p, $r \models r$, q and $r \models r$.

(k) So p, NK$pr \models$ Kpq, KNqNr (f), and by (c), (d), and (e), KpNKNqNr $\models$ NKNKpqNKpr. QED.

(l) Other direction: NKNKpqNK$pr \models$ KpNKNqNr.

(m) NKNKpqNK$pr \models$ KpNKNqNr iff
  (i) NKNKpqNK$pr \models p$ and
  (ii) NKNKpqNK$pr \models$ NKNqNr (RK).

(n) Unpack (m–i): NKNKpqNK$pr \models p$ iff $\models p$, KNKpqNKpr (LN) iff
  (i) $\models p$, NKpq and
  (ii) $\models p$, NKpr.

(o) Unpack (n–i): $\models p$, NKpq iff K$pq \models p$ (RN). So (n–i) holds.

(p) Unpack (n–ii): $\models p$, NKpr iff K$pr \models p$ (RN). So (n–ii) holds.

(q) So (m–i) holds.

(r) Unpack (m–ii): NKNKpqNK$pr \models$ NKNqNr iff $\models$ KNKpqNKpr, NKNqNr (LN).

(s) $\models$ KNKpqNKpr, NKNqNr iff KNqN$r \models$ KNKpqNKpr (RN).

(t) KNqN$r \models$ KNKpqNKpr iff
  (i) KNqN$r \models$ NKpq and
  (ii) KNqN$r \models$ NKpr.

(u) Unpack (t–i): KNqN$r \models$ NKpq iff KNqNr, K$pq \models \emptyset$ (RN).

(v) KNqNr, K$pq \models \emptyset$ iff Nq, Nr, K$pq \models \emptyset$ (LK) iff Nq, K$pq \models r$ (LN) iff K$pq \models r$, q (LN). But K$pq \models r$, q because K$pq \models q$, since $q \models q$, by (LK). So (t–i) holds.

(w) Unpack (t–ii): KNqN$r \models$ NKpr iff KNqNr, K$pr \models \emptyset$ (RN) iff Nq, Nr, K$pr \models \emptyset$ (LK) iff Nr, K$pr \models q$ (LN) iff K$pr \models q$, r (LN). But K$pr \models q$, r iff p, $r \models q$, r (LK), and p, $r \models q$, r because $r \models r$. So (t–ii) holds.

(x) So, by (t): KNqN$r \models$ KNKpqNKpr, so (m–ii) holds.

(y) By (x) and (q), (m) holds: NKNKpqNK$pr \models$ KpNKNqNr. QED.

(z) So by (y) and (k): KpNKNqN$r \approx$ NKNKpqNKpr. So KpA$qr \approx$ AKpqKpr, and distributivity holds. QED.

It follows that N, K behave entirely classically.

Besides A, it is useful to define the conditional $\mathrm{C}pq \approx_{\mathrm{df}} \mathrm{NK}p\mathrm{N}q$. A conditional is valid just in case the corresponding entailment holds:

3.3 $p \vDash q \Leftrightarrow\ \vDash \mathrm{C}pq$.

Proof: By the definition of C, $\vDash \mathrm{C}pq \Leftrightarrow\ \vDash \mathrm{NK}p\mathrm{N}q$. By RN, $\vDash \mathrm{NK}p\mathrm{N}q \Leftrightarrow \mathrm{K}p\mathrm{N}q \vDash \emptyset$. By LK, $\mathrm{K}p\mathrm{N}q \vDash \emptyset \Leftrightarrow p, \mathrm{N}q \vDash \emptyset$. By LN, $p, \mathrm{N}q \vDash \emptyset \Leftrightarrow p \vDash q$.

Given 3.3, contraposition across $\vDash (p \vDash q \Leftrightarrow \mathrm{N}q \vDash \mathrm{N}p)$, ~~proven~~ in 1.3, proved
has as an immediate consequence contraposition for C:

3.4 (Contraposition 2) $\vDash \mathrm{C}pq \Leftrightarrow\ \vDash \mathrm{CN}q\mathrm{N}p$.

We get detachment (*modus ponens*) as a derived rule:

3.5 (Detachment) $\mathrm{KC}pqp \vDash q$.

Proof: By LK, $\mathrm{KC}pqp \vDash q \Leftrightarrow \mathrm{C}pq, p \vDash q$. By the definition of C, $\mathrm{C}pq, p \vDash q \Leftrightarrow \mathrm{NK}p\mathrm{N}q, p \vDash q$. By LN, $\mathrm{NK}p\mathrm{N}q, p \vDash q \Leftrightarrow p \vDash \mathrm{K}p\mathrm{N}q, q$. By RK, $p \vDash \mathrm{K}p\mathrm{N}q, q \Leftrightarrow (p \vDash p, q$ and $p \vDash \mathrm{N}q, q)$. The first of these hold by Weakening, and the second holds since $p \vDash \mathrm{N}q, q$ is equivalent to $p, q \vDash q$ (LN), and $q \vDash q$.

The following two results may help to impart a better intuitive grasp of the behavior of the connectives. The first vindicates the heuristic reading of "$X \vDash p, q$" as "X entails p or q."

3.6 $X \vDash \mathrm{A}pq \Leftrightarrow X \vDash p, q$.

Proof: $X \vDash \mathrm{A}pq \Leftrightarrow X \vDash \mathrm{NKN}p\mathrm{N}q$ (definition) $\Leftrightarrow X, \mathrm{KN}p\mathrm{N}q \vDash \emptyset$ (RN) $\Leftrightarrow X, \mathrm{N}p, \mathrm{N}q \vDash \emptyset$ (LK) $\Leftrightarrow X \vDash p, q$ (LN).

The next result generalizes Negation Introduction. Where the latter claims that X is incompatible with $\{\mathrm{N}p\}$ just in case X entails p, we now show how X relates to multiple negated formulas. In essence, X is incompatible with $\{\mathrm{N}p_1, \ldots, \mathrm{N}p_n\}$ just in case X entails (p_1 or … or p_n).

3.7 $X \cup \{\mathrm{N}p_1, \ldots, \mathrm{N}p_{n\}} \in \mathrm{Inc}$ iff $X \vDash p_1, \ldots, p_n$.

Proof: The claim is equivalent to $X, \mathrm{N}p_1, \ldots, \mathrm{N}p_n \vDash \emptyset$ iff $X \vDash p_1, \ldots, p_n$. This latter claim follows by n applications of LN.

4 Modality

The *K* axiom is validated by the incompatibility semantics:

4.1 (K) $\models$ CLCpqCLpLq.

Proof: If we are in the degenerate frame, the result follows. So assume instead that $\emptyset \not\models \emptyset$.

(a) Since we have already shown in 3.3 that $p \models q \Leftrightarrow \models$ Cpq, it suffices to show LC$pq \models$ CLpLq.
(b) By RL, LC$pq \models$ CLpL$q \Leftrightarrow$ ($\models$ CLpLq or $\not\models$ Cpq).
(c) By 3.3, $p \models q \Leftrightarrow \models$ Cpq, $\models$ CLpL$q \Leftrightarrow$ L$p \models$ Lq.
(d) By RL, L$p \models$ L$q \Leftrightarrow$ L$p \models \emptyset$ or $\models q$.
(e) By LL, (L$p \models \emptyset$ or $\models q$) $\Leftrightarrow$ ($\emptyset \models \emptyset$ or $\not\models p$ or $\models q$) $\Leftrightarrow$ ($\not\models p$ or $\models q$).
(f) Since $p \models q \Leftrightarrow \models$ Cpq, $\not\models$ C$pq \Leftrightarrow p \not\models q$.
(g) So LC$pq \models$ CLpL$q \Leftrightarrow$ ($\not\models p$ or $\models q$ or $p \not\models q$).
(h) Suppose not. Then ($\models p$ and $\not\models q$ and $p \models q$). But we showed in 1.1.2 of Appendix 1 that if $p \models q$ and $\models p$, then $\models q$. So this is a contradiction. So LC$pq \models$ CLpLq.

Since PC with *modus ponens* (and substitution) is validated, and we showed in 3.4 of Appendix 1 that the rule of necessitation $\models p \Rightarrow \models$ Lp holds, so is the minimal modal system K.

From this it is easy to show that the T axiom—and hence the modal system T—is validated:

4.2 (T) $\models$ CLpp.

Proof: We also showed in 3.5 of Appendix 1 that L$p \models p$, and in 3.3 that $p \models q \Leftrightarrow \models$ Cpq.

4.3 (S4) $\models$ CLpLLp.

Proof:

(a) By 3.3, $\models$ CLpLL$p \Leftrightarrow$ L$p \models$ LLp.
(b) By RL, L$p \models$ LL$p \Leftrightarrow$ L$p \models \emptyset$ or $\models$ Lp.
(c) By LL, L$p \models \emptyset \Leftrightarrow \emptyset \models \emptyset$ or $\not\models p$.
(d) By RL, $\emptyset \models$ L$p \Leftrightarrow \emptyset \models \emptyset$ or $\models p$.

(e) So, plugging (b) and (c) into (a): L$p \models$ LL$p \Leftrightarrow (\emptyset \models \emptyset$ or $\not\models p$ or $\emptyset \models \emptyset$ or $\models p)$.
(f) So L$p \models$ LL$p \Leftrightarrow \emptyset \models \emptyset$ or $(\not\models p$ or $\models p)$. But this second disjunct always holds.

Since the system S4 is just T plus the S4 axiom, the incompatibility semantics validates S4.

4.4 (S5) $\models$ CMpLMp.

Proof:

(a) By 3.3, $\models$ CMpLM$p \Leftrightarrow$ M$p \models$ LMp.
(b) M$p \models$ LM$p \Leftrightarrow$ NLN$p \models$ LNLNp, since M$p \approx$ NLNp.
(c) By LN, NLN$p \models$ LNLNp iff $\models$ LNLNp, LNp.
(d) By RL, $\models$ LNLNp, LN$p \Leftrightarrow \models$ LNLNp or $\models$ Np.
(e) By RN, $\models$ N$p \Leftrightarrow p \models \emptyset$.
(f) So, plugging (d) into (c) and (c) into (b): NLN$p \models$ LNLN$p \Leftrightarrow \models$ LNLNp or $p \models \emptyset$.
(g) By RL, $\emptyset \models$ LNLN$p \Leftrightarrow \emptyset \models \emptyset$ or $\models$ NLNp.
(h) By RN, $\models$ NLN$p \Leftrightarrow$ LN$p \models \emptyset$.
(i) By LL, LN$p \models \emptyset \Leftrightarrow \emptyset \models \emptyset$ or $\not\models$ Np.
(j) By RN, $\models$ N$p \Leftrightarrow p \models \emptyset$, so $\not\models$ N$p \Leftrightarrow p \not\models \emptyset$.
(k) Plugging (i) into (h) and (h) into (g): $\models$ NLN$p \Leftrightarrow \emptyset \models \emptyset$ or $p \not\models \emptyset$.
(l) Plugging (g) into (f) and (f) into (e): NLN$p \models$ LNLN$p \Leftrightarrow \emptyset \models \emptyset$ or $\emptyset \models \emptyset$ or $p \not\models \emptyset$ or $p \models \emptyset$.
(m) But $p \not\models \emptyset$ or $p \models \emptyset$ always holds, so NLN$p \models$ LNLNp always holds.

So the incompatibility semantics validates S5.

Appendix 3: Basic semantic results in the metatheory of incompatibility logic[13]

The proof of completeness follows the familiar Henkin route. Our primary result is that any S5-consistent set Z is satisfiable and we show this by extending Z to a maximal consistent set Z^* and reading a model off of Z^*. But there are some twists because of the context in which

[13] These proofs are due to Alp Aker.

we are working. The relevant notion of satisfaction is not truth in a model, but rather coherence on a frame. So we do not use Z^* to tell us which sentences are true, but instead use it to tell us which sets of sentences are incoherent; that is, we are looking to find out which sets of sentences are to be considered materially inconsistent. Further, our notion of incoherence is a modal notion, so we are interested in which sets of sentences are *necessarily* materially inconsistent. Thus instead of taking each member of Z^* to assert a truth in the model we construct, we consider only members of Z^* of the form $\mathrm{LNK}x_1\mathrm{K}x_2\ldots\mathrm{K}x_{n-1}x_n$ and we take such sentences to specify the incoherent sets of the desired frame.

1 Consistent Sets Are Satisfiable

1.1 Let Z be a set of S5-consistent sentences in the language L (a set such that it is not the case that $Z \vdash_{S5} \mathrm{K}p\mathrm{N}p$). Then there is a frame Inc_Z such that Z is coherent on Inc_Z (that is, $Z \notin \mathrm{Inc}_Z$).

Proof: Let Z^* be a maximal consistent superset of Z, and hence deductively closed. By the familiar proof of the Lindenbaum Lemma, on the assumption that Z is consistent, it is easily verified that Z^* exists and that, for each p, either p or $\mathrm{N}p$ is in Z^*.

We introduce some new notational conventions for the sake of clarity. Let $\mathrm{K}X$ be the conjunction of all x_i in X, that is, $\mathrm{K}X$ abbreviates $\mathrm{K}x_1\mathrm{K}x_2\ldots\mathrm{K}x_{n-1}x_n$. (If $X = \{p\}$ let $\mathrm{K}X = p$.)

We define our desired frame Inc_Z as follows: $X \in \mathrm{Inc}_Z$ iff there is some finite subset $X' \subseteq X$ such that $\mathrm{LNK}X' \in Z^*$.

Note that because conjunction is associative and commutative we can neglect the precise ordering and nesting of the conjuncts in $\mathrm{K}X'$ when considering $\mathrm{LNK}X' \in Z^*$. In the sequel we use this fact without notice.

1.2 Inc_Z satisfies the frame axioms.

Proof: (Persistence) Immediate from construction.

(KI) $X \cup \{\mathrm{K}pq\} \in \mathrm{Inc}_Z$ iff $X \cup \{p, q\} \in \mathrm{Inc}_Z$.

Suppose $X \cup \{\mathrm{K}pq\} \in \mathrm{Inc}_Z$. If that is so then there is a $X' \subseteq X$ such that for some ordering on X we have $\mathrm{LNKK}X'\mathrm{K}pq \in Z^*$.

It follows that $X \cup \{p, q\} \in \text{Inc}_Z$. The converse is proved in the same fashion.

(NI) X, N$p \in \text{Inc}_Z$ iff $X \models p$.

Suppose $X \cup \{\text{N}p\} \in \text{Inc}_Z$. We want $X \models p$. So assume $W \cup \{p\} \in \text{Inc}_Z$. We want $W \cup X \in \text{Inc}_Z$. Since $W \cup \{p\} \in \text{Inc}_Z$ there is some finite subset $W' \subseteq W$ such that LNKK$W'p \in Z^*$. Since $X \cup \{\text{N}p\} \in \text{Inc}_Z$ we know there is some finite $X' \subseteq X$ such that LNKKX'N$p \in Z^*$. Now, NKK$W'p$ and NKKX'Np truth-functionally imply NKKX'KW', so LNKKX'K$W' \in Z^*$, which means $W \cup X \in \text{Inc}_Z$.

Now suppose $X \models p$. We want $X \cup \{\text{N}p\} \in \text{Inc}_Z$. Now, we know LNK$pNp$ is an S5-theorem and so is in Z^*. Thus $\{p, \text{N}p\} \in \text{Inc}_Z$. Since $X \models p$, we then know $X \cup \{\text{N}p\} \in \text{Inc}_Z$.

(LI) $X \cup \{\text{L}p\} \in \text{Inc}_Z$ iff $X \in \text{Inc}_Z$ or $\exists Y(X \cup Y \notin \text{Inc}_Z$ & $Y \not\models p)$.

Suppose $X \cup \{\text{L}p\} \in \text{Inc}_Z$. We want $X \in \text{Inc}_Z$ or $\exists Y(X \cup Y \notin \text{Inc}_Z$ & $Y \not\models p)$. We show the equivalent claim that if $\forall Y(X \cup Y \in \text{Inc}_Z$ or $Y \models p)$, then $X \in \text{Inc}_Z$.

Assume the antecedent and instantiate {NLp} for Y. $X \cup \{\text{NL}p\} \in \text{Inc}_Z$ or NL$p \models p$. We want to show $X \in \text{Inc}_Z$ for each disjunct.

By the first disjunct and our supposition, we have, unpacking the definitions: there is some finite $X' \subseteq X$ such that LNKKX'L$p \in Z^*$ and LNKKX'NL$p \in Z^*$. Since NKKX'Lp and NKKX'NLp truth-functionally imply NKX', it follows that LNKX' and so $X \in \text{Inc}_Z$.

Applying NI to our second disjunct gives us $\{\text{NL}p, \text{N}p\} \in \text{Inc}_Z$. Then LNKNL$pNp \in Z^*$, which is equivalent to LCNL$pp \in Z^*$. This, together with LCLpp, gives us Lp (since CLpp and CNLpp truth-functionally imply p), which by S4 gives us LLp. Unpacking our supposition $X \cup \{\text{L}p\} \in \text{Inc}_Z$ gives us LNKKX'L$p \in Z^*$, which, with LLp, implies LNK$X' \in Z^*$ (since Lp and NKKX'Lp truth-functionally imply NKX'). Thus $X \in \text{Inc}_Z$.

Suppose $X \in \text{Inc}_Z$ or $\exists Y(X \cup Y \notin \text{Inc}_Z$ & $Y \not\models p)$. We want $X \cup \{\text{L}p\} \in \text{Inc}_Z$ for each disjunct. If $X \in \text{Inc}_Z$, it follows immediately that $X \cup \{\text{L}p\} \in \text{Inc}_Z$. If $\exists Y(X \cup Y \notin \text{Inc}_Z$ & $Y \not\models p)$, then

$\not\models p$ and by NI N$p \notin \text{Inc}_Z$. Then L$p \notin Z^*$. By maximality of Z^*, NL$p \in Z^*$, which implies LNL$p \in Z^*$ by the S5 axioms. Then L$p \in \text{Inc}_Z$ and our result follows immediately. Thus Inc_Z as we have defined it is a frame. QED (1.2).

1.3 Our original Z is coherent in Inc_Z.

Proof: If Z is incoherent then there is some finite $Z' \subseteq Z$ such that LNK$Z' \in Z^*$ and so NK$Z' \in Z^*$. But $z_i' \in Z^*$ for each $z_j' \in Z'$, and so K$Z' \in Z^*$, which is a contradiction. QED (1.3).

With 1.3 the proof of 1.1 is complete.

2 Completeness of S5 with respect to the incompatibility semantics

2.1. Suppose $\models_{\text{Inc}} p$ on every frame Inc. Then $\vdash_{\text{S5}} p$.

Proof: If $\models_{\text{Inc}} p$ on every Inc, then N$p \models_{\text{Inc}} \emptyset$ on every Inc, and so by 1.1 it follows that Np is S5-inconsistent, in which case $\vdash_{\text{S5}} p$.

This generalizes in the familiar way:

2.2. Suppose $X \models_{\text{Inc}} p$ on every frame Inc. Then $X \vdash_{\text{S5}} p$.

Proof: If $X \models_{\text{Inc}} p$ on every Inc, then $X \cup \{\text{N}p\} \in$ Inc for every Inc, and so by 1.1 we know $X \cup \{\text{N}p\}$ is S5-inconsistent, and so $X \vdash_{\text{S5}} p$.

3 Soundness

We adopt as a convenient formulation of the proof theory of S5 the following sequent calculus.

Axioms:

(K) $\vdash$ CLCpqCLpLq
(T) $\vdash$ CLpp
(S5) $\vdash$ CNLNpLNLNp

Rules:

(Identity) $p \vdash p$
(Contraction) $V, X, X, Y \vdash Z \Rightarrow V, X, Y \vdash Z$
(Weakening) $X, V, Y \vdash Z \Rightarrow X, W, V, Y \vdash Z$
(Permutation) $X, V, W, Y \vdash Z \Rightarrow X, W, V, Y \vdash Z$
($\vdash$ N) $X, p \vdash Z \Leftrightarrow X \vdash Z, \text{N}p$

(N $\vdash$) $X \vdash p, Z \Leftrightarrow X, \mathrm{N}p \vdash Z$
($\vdash$ K) $X \vdash p, Z$ and $X \vdash q, Z \Leftrightarrow X \vdash \mathrm{K}pq, Z$
(K $\vdash$) $X, p, q \vdash Z \Leftrightarrow X, \mathrm{K}pq \vdash Z$
(Cut) $X \vdash p, Y$ and $V, p \vdash W \Rightarrow X, V \vdash Y, W$
(Necessitation) $\vdash p \Rightarrow \vdash \mathrm{L}p$

3.1 If $X \vdash_{S5} p$, then $X \models p$ on all frames.

Proof: By induction on proof complexity. That is, we verify that each of the axioms is validated by the incompatibility semantics and that each rule preserves validity. The verifications of the rules can be found in Appendix 1 (see 2.1, 4.1.2, 4.1.1, 4.1.4, 4.1.3, 2.2, and 3.4). The verifications of the axioms can be found in Appendix 2 (see 4.1, 4.2, and 4.4).

3.2 If X is S5-inconsistent, then X is incoherent on all frames.

Proof: If X is S5-inconsistent, then there is some finite $X' \subseteq X$ such that $X' \vdash_{S5} p$ and $X' \vdash_{S5} \mathrm{N}p$. But then $X' \models p$ and $X' \models \mathrm{N}p$ on all frames. But then $X' \cup \{\mathrm{N}p\} \in \mathrm{Inc}$ and $X' \models \mathrm{N}p$ on all frames, so $X' \in \mathrm{Inc}$ for all Inc. Thus $X \in \mathrm{Inc}$ for all Inc.

4 Compactness

4.1 If for every finite $Z' \subseteq Z$ there is a frame $\mathrm{Inc}_{Z'}$ on which Z' is coherent, then there is a frame Inc_Z on which Z is coherent.

Proof: Suppose not. If Z is not coherent on any frame then by 1.1 it is S5-inconsistent. Then it has some S5-inconsistent subset Z′. By 3.2 Z′ is incoherent on frames, which is a contradiction.

Appendix 4: Representation of consequence relations by incompatibility relations

1 Imputing incompatibility relations from consequence relations[14]

1.1 Preliminary remarks We assume that we have a consequence relation $\vdash$ whose consequent position is either empty or filled by a single sentence.

[14] The original representation theorem was proved by the author. But it has been substantially sharpened, and the proof improved, by Alp Aker. Besides the Defeasibility condition required for completeness, the first proof appealed to four conditions that were sufficient for soundness:

That is, $\vdash$ is a relation between sets of sentences and single sentences (or the empty set) for some language L. (We consider other types of consequence relations below; see section 6.) This may be a *material* consequence relation, if the sentences do not have any internal logical complexity (or if we are ignoring what they do have), or it may be a *logical* consequence relation, perhaps defined axiomatically, or by a natural deduction system, or by a sequent calculus.

The Representation Theorem for turnstile $\vdash$ has two conditions:

- General Transitivity: $\forall X, Y \subseteq L\ \forall p, q \in L[(X \vdash p \ \&\ \{p\} \cup Y \vdash q) \rightarrow X \cup Y \vdash q]$.
- Defeasibility: $\forall X \subseteq L\, \forall p \in L[\sim(X \vdash p) \rightarrow \exists Y \subseteq L[\forall q \in L[\{p\} \cup Y \vdash q] \ \&\ \exists q \in L[\sim(X \cup Y \vdash q)]]]$.

Note that General Transitivity has Pure Transitivity as a special case, where Pure Transitivity is:

- Pure Transitivity: $\forall X \subseteq L \forall p, q \in L[(X \vdash p \ \&\ \{p\} \vdash q) \rightarrow X \vdash q]$.

We simply take $Y = \emptyset$.

1.2 *Representation definitions*

(i) $\text{Inc}(X)$ iff $\forall p \in L[X \vdash p]$.

The basic idea is to read off an incoherence relation from the consequence relation by taking the incoherent sets to be the ones that have *everything* as their consequence. If we start with a *logical* consequence relation, generated by a logic that has *ex falso quodlibet* as a basic or derived rule, this will just be the inconsistent sets: the ones that have as a consequence some sentence and its negation.

We then define incompatibility from incoherence in the usual way:

(ii) $X \in \text{I}(Y)$ iff $\text{Inc}(X \cup Y)$.

(i) Reflexivity: $\forall X \subseteq L[X \vdash X]$.
(ii) Transitivity: $\forall X, Y, Z \subseteq L[(X \vdash Y \ \&\ Y \vdash Z) \rightarrow X \vdash Z]$.
(iii) Monotonicity: $\forall X, Y, Z \subseteq L[(X \vdash Y \ \&\ X \subseteq Z) \rightarrow Z \vdash Y]$.
(iv) Amalgamation: $\forall X, Y, Z \subseteq L[(X \vdash Y \ \&\ X \vdash Z) \rightarrow X \vdash Y \cup Z]$.

Aker showed that, although these conditions are indeed sufficient for the imputed incompatibility relation to generate a semantic consequence relation $\models$ that would hold whenever the original consequence relation $\vdash$ did, they were not in fact necessary for that result. He showed further that General Transitivity is both necessary and sufficient. The "Converse Results" presented below are also due to Aker.

And also define the incompatibility-consequence relation as usual:

(iii) $X \models_I p$ iff $\forall Z \subseteq L[Z \in I(p) \rightarrow Z \in I(X)]$.

1.3 Soundness and completeness

Representation Theorem $\vdash$ is sound and complete with respect to $\models_I$ if, and only if, $\vdash$ satisfies General Transitivity and Defeasibility.

We first give the proof from right to left. That is, we show soundness and completeness assuming General Transitivity and Defeasibility.

3.1 (Soundness) If $X \vdash p$, then $X \models p$.

Proof Suppose not. Then $X \vdash p$ but not $X \models p$ for some X and p. Then by definition of $\models$ there is some $Z \in I(p)$ while $Z \notin I(X)$. Unpacking definitions we have $\forall q \in L[\{p\} \cup Z \vdash q]$ and $\exists r \in L[\sim(X \cup Z \vdash r)]$. Choose some such witnessing r so that $\sim(X \cup Z \vdash r)$. Instantiating $\forall q \in L[\{p\} \cup Z \vdash q]$ we also know $\{p\} \cup Z \vdash r$. Since $X \vdash p$, it follows from General Transitivity that $X \cup Z \vdash r$, which is a contradiction.

3.2 (Completeness) If $X \models p$, then $X \vdash p$.

Proof: Suppose not. Since $\sim(X \vdash p)$ we know by Defeasibility that there are V and r such that $\forall q \in L[\{p\} \cup V \vdash q]$ and $\sim(X \cup V \vdash r)$. Since $X \models p$ we have $\forall Z \subseteq L[Z \in I(p) \rightarrow Z \in I(X)]$. Unpacking the definition further we have $\forall Z \subseteq L[\forall q \in L(\{p\} \cup Z \vdash q) \rightarrow \forall q \in L(X \cup Z \vdash)]$. Instantiating with $Z = V$ we have $\forall q \in L(\{p\} \cup V \vdash q) \rightarrow \forall q \in L(X \cup V \vdash q)$. By modus ponens we have $\forall q \in L(X \cup V \vdash q)$. Instantiating with $q = r$ we have $X \cup V \vdash r$, which contradicts $\sim(X \cup V \vdash r)$.

1.4 Converse results We now show that $\vdash$ satisfies General Transitivity and Defeasibility, assuming $\vdash$ is sound and complete with respect to $\models$.

4.1 (General Transitivity) $\forall X, Y \subseteq L\, \forall p, q \in L[(X \vdash p \,\&\, \{p\} \cup Y \vdash q) \rightarrow X \cup Y \vdash q]$.

Proof: Suppose $X \vdash p$ and $\{p\} \cup Y \vdash q$. By soundness $X \models p$ and $\{p\} \cup Y \models q$. We show $X \cup Y \models q$. To show this, we need to show that $V \in \mathrm{I}(q)$ implies $V \in \mathrm{I}(X \cup Y)$. So suppose $V \in \mathrm{I}(q)$. Since $\{p\} \cup Y \models q$, this implies that $V \in \mathrm{I}(\{p\} \cup Y)$, which implies $\mathrm{Inc}(V \cup \{p\} \cup Y)$, which in turn implies $V \cup Y \in \mathrm{I}(p)$. Since $X \models p$, we then know that $V \cup Y \in \mathrm{I}(X)$. This is equivalent to $\mathrm{Inc}(V \cup X \cup Y)$, which is in turn equivalent to $V \in \mathrm{I}(X \cup Y)$. Hence $X \cup Y \models q$. By completeness, $X \cup Y \vdash q$.

4.2 (Defeasibility) $\forall X \subseteq L\ \forall p \in L[\sim(X \vdash p) \rightarrow \exists Y \subseteq L[\forall q \in L[\{p\} \cup Y \vdash q]\ \&\ \exists q \in L[\sim(X \cup Y \vdash q)]]]$.

Proof: Suppose $\sim(X \vdash p)$. By completeness, $\sim(X \models p)$. Unpacking the definition of $\models$, we have $\exists Y[Y \in \mathrm{I}(p)\ \&\ \sim(Y \in \mathrm{I}(X))]$. Unpacking the definitions of $Y \in \mathrm{I}(p)$ and $\sim(Y \in \mathrm{I}(X))$, we have $\forall q \in L[\{p\} \cup Y \vdash q]\ \&\ \exists q \in L[\sim(X \cup Y \vdash q)]$.

1.5 Discussion We have shown that General Transitivity and Defeasibility are jointly equivalent to soundness and completeness. As noted, this initially looks like an ideal result. But the reader might have noticed that our proofs allow for a more precise characterization of the logical relation between these four properties. Put briefly, the situation is this:

Completeness if, and only if, Defeasibility.

That is, we appealed only to Defeasibility in the proof of completeness, and vice versa. One might expect, then, that we would have:

Soundness if, and only if, General Transitivity.

But a look at the proofs reveals instead that we have:

General Transitivity implies soundness.
Soundness and completeness imply General Transitivity.

We have not been able to eliminate an appeal to completeness in the proof of General Transitivity.

1.6 Generalizations Having identified the incoherent sets and the semantic entailments, we could proceed to reason logically in the language L. The rules of incompatibility logic are not directly applicable because those rules in general require that the consequents of $\models$ can be sets of formulae, not just single sentences. But having identified the incoherent sets, the entailment relation we have been using in previous sections is perfectly well-defined. Of course, when we allow claims such as $X \models Y$ we will not have a corresponding consequence $X \vdash Y$ because of expressive limits on $\vdash$.

In cases where $\vdash$ is more expressive, in the sense of allowing multiple formulae in consequent position, there is still a representation result, but the conditions must be slightly different. If $X \vdash \{y_1, \ldots, y_n\}$ has the meaning "X implies y_1 and ... and y_1" then the conditions for representation are:

- General Transitivity: $\forall X, Y, W, Z \subseteq L[(X \vdash Y \; \& \; Y \cup W \vdash Z) \rightarrow X \cup W \vdash Z]$.
- Defeasibility: $\forall X, Y \subseteq L[\sim(X \vdash Y) \rightarrow \exists Z \subseteq L[\forall W \subseteq L[Y \cup Z \vdash W] \; \& \; \exists W \subseteq L[\sim(X \cup Z \vdash W)]]]$.

And we must, naturally, also adjust our representation definitions:

(i) $\mathrm{Inc}(X)$ iff $\forall U \subseteq L[X \vdash U]$.
(ii) $X \in \mathrm{I}(Y)$ iff $\mathrm{Inc}(X \cup Y)$.
(iii) $X \models_{\mathrm{I}} Y$ iff $\forall Z \subseteq L[Z \in \mathrm{I}(Y) \rightarrow Z \in \mathrm{I}(X)]$.

With these changes the representation theorem again holds. Indeed, the proofs require only trivial modification.

If $\vdash$ is instead a disjunctive-consequent turnstile (that is, $X \vdash \{y_1, \ldots, y_n\}$ has the meaning "X implies y_1 or ... or y_1"), then the conditions are again different. We can retain Defeasibility as in the previous case, but our transitivity condition becomes:

- General Transitivity: $\forall X, W, Z \subseteq L \;\; \forall y_1, \ldots, y_n \in L[(X \vdash y_1, \ldots, y_n \; \& \; \{y_1\} \cup W \vdash Z \; \& \ldots \& \; \{y_n\} \cup W \vdash Z) \rightarrow X \cup W \vdash Z]$.

The representation definitions are also as in the previous case, except for the definition of entailment:

(iii) $X \models_{\mathrm{I}} Y$ iff $\forall Z \subseteq L[Z \in \bigcap_{p \in Y} \mathrm{I}(p) \rightarrow Z \in \mathrm{I}(X)]$.

The proofs again require only obvious changes.

2 *Discussion of some logical consequence relations*

All the logics we consider satisfy General Transitivity. For, as pointed out in the text, that condition is equivalent to the Cut rule:

$$\frac{\Gamma : A \text{ and } \Delta, A : B}{\Gamma, \Delta : B}.$$

This will hold as a derived rule in any system that permits the argument:

$$\frac{\Delta, A : B}{\Delta : A \rightarrow B}$$

by '→' Intro, and then:

$$\frac{\Gamma : A \text{ and } \Delta : A \rightarrow B}{\Gamma, \Delta : B}$$

by '→' Elimination.

And all except relevance logics include *ex falso quodlibet*, equivalent to the rule:

$$\frac{\Gamma : A \text{ and } \Gamma : {\sim}A}{\Gamma : B}$$

which is required to identify incoherent sets on the basis of their role in the consequence relation, so as to impute the incompatibility relation which in turn determines the incompatibility-entailments.

So the significant condition to consider with respect to various logical consequence relations is Defeasibility.

2.1 Classical logic It is easy to show that classical logic *does* satisfy Defeasibility. For in this context, defeasibility just comes to the condition that if it is not the case that $p \vdash q$, then there is something that is *inconsistent* with with q and not with p. If q is not a consequence of p, there must be some interpretation on which p is true and q is not true. Pick one such. Now it might, or it might not, be the case that p and q are incompatible or inconsistent (that is, that $\forall U[\{p\} \cup \{q\} \vdash U]$). If they are *not* incompatible, then ${\sim}q$ is incompatible with q (that is $\forall U[\{q\} \cup \{{\sim}q\} \vdash U]$) and not with p. If p and q *are* incompatible, then p itself is something that is incompatible with q and not with p—unless p were itself incoherent (= inconsistent: $\forall U[p \vdash U]$), in which case $p \vdash q$, contrary to our hypothesis. So if q is not a logical consequence of p, then there is something that is incompatible with q and not with p, which is the defeasibility condition.

2.2 Modal logics Most familiar modal logics, including T (sometimes called 'M'), K, B, S4 and S5 (indeed, all the Lewis systems), and many less familiar ones (such as Boolos's GL modal logic of provability) contain all the theorems of the classical propositional calculus PC.

Defeasibility and the arguments and constructions concerning it depend only on the effects of classical negation on the logical consequence relation, so they go through straightforwardly for all normal modal logics.

2.3 Intuitionism The consequence relation of intuitionistic logic does not satisfy defeasibility. It is the case that whenever an intuitionistic consequence is a good one, everything incompatible (here, inconsistent) with the consequent is incompatible with the antecedent. (That much follows from the soundness result, which depends only on Cut.) But it is not the case that wherever an intuitionistic consequence *fails* there is something that is inconsistent with the consequent but not the antecedent. For instance, it is characteristic of intuitionism that although $\neg\neg p$ does follow from p, p is not a consequence of $\neg\neg p$. The Defeasibility condition requires that there be a 'witness' of the badness of this inference, in the form of something incompatible with p, but not with $\neg\neg p$. In this setting, what is incompatible with p is what is inconsistent with it, and that is whatever entails $\neg p$. But everything that entails $\neg p$ is inconsistent *both* with p *and* with $\neg\neg p$. So there can be no such witness. So Defeasibility fails for the consequence relation of intuitionistic logic. Indeed, the cases where it fails, the non-consequences that fail to have the witnesses incompatibility-defeasibility demands, are just those classical inferences that do not hold good in the intuitionist setting. So intuitionism can be characterized precisely by the cases in which incompatibility-defeasibility fails.

Since the second condition of the representation theorem proved above does not hold for intuitionism, the intuitionistic logical consequence relation is not fully captured by the incompatibility-consequence relation implicit in it. Does that mean that the intuitionistic propositional calculus (and its modal extensions such as intuitionistic S4) does *not* have PC + S5 as its consequence-intrinsic logic? That conclusion would be hasty. For the intuitionistic notion of negation defines a notion of inconsistency that when made to play the role of incompatibility generates a standard incompatibility-consequence relation: that is, one whose proper elaborated-explicating (LX) implicit logical vocabulary is PC + S5. It follows that the techniques introduced here show that alongside the logical consequence relation explicitly defined by the axioms, natural deduction rules, or sequents of intuitionistic propositional calculus, there is *another* logical consequence relation implicitly put in play by the relation of intuitionistic inconsistency defined by intuitionistic negation. Defeasibility *does* hold for that one, and it permits the introduction of the classical connectives plus the S5 modal connectives, by the means outlined in these appendices. In this somewhat extended sense, PC + S5 *is* the intrinsic logic of intuitionism (and its modal extensions such as intuitionistic S4) too.

To these considerations we may add another, which may be instructive in comparative assessments of the expressive power of intuitionistic versus classical logical connectives (the issue that supersedes concern over which is the *true* or *correct* logic, on the expressive view of the demarcation of logical vocabulary pursued here). If we look at small finite numbers of propositions—say n atomic propositions, along with some, but not all of their negations, and some, but not all of the conditionals relating them—it will often happen that for some incompatibility interpretations (even those that respect the meanings of the connectives to the extent possible, for instance by ensuring that any set containing $\sim p$ is incompatible with any containing p), some inferences we take to be bad ones are endorsed, because everything incompatible with the consequent is incompatible with the antecedent. Intuitively, this is because there just are not *enough* propositions—not enough, that is, to provide witnesses, incompatibility-defeasors, for all the bad inferences. Throwing in some more propositions, for instance, adding more negations, more conditionals, negations of conditionals, and so on, provides the desired defeasors. As n gets larger, and as we more completely form the logical compounds of those atomic propositions, the incompatibility-consequence relation converges on the intuitively—and logically—correct one. One might think of the situation with the two consequence relations generated by intuitionistic logic—the one it defines directly and the one generated by its notion of inconsistency—along these lines. The intuitionistic consequence relation tells us that some consequences are bad, that they do not hold: paradigmatically, the inference from $\neg\neg p$ to p. But while for *most* of the consequences that fail in the intuitionistic setting (for instance, that from $p \vee q$ to $p \,\&\, q$) it is possible to give *reasons* justifying the claim that the inference is a bad one, in the form of inconsistency-defeasors, sets of claims that are inconsistent with the conclusion but not with the premises, for *some* (indeed, for just those whose failure distinguishes intuitionistic from classical logic), it is *not* possible to formulate such defeasors, to give reasons of that kind. From the incompatibility point of view—and keeping in mind the way failures to yield incompatibility-defeasors for intuitively bad inferences can be seen to be due to the expressive impoverishment of systems with "too few" propositions—the failure of Defeasibility for what we may call the 'official' consequence relation of intuitionistic logic amounts to an admission of expressive impoverishment. The intuitionistic logical vocabulary does not have the expressive power to formulate defeasors that could serve as witnesses, as reasons for denying the goodness of inferences the logic nonetheless insists are bad. Such reasons can be given for some of the inferences it rejects (those that are rejected also by classical logic), but not for the rest.

Again, from this point of view, intuitionistic logic shows itself to be incomplete. To defeasor-complete a system containing intuitionistic negation, one would want to add another kind of negation, so contrived that it would supply defeating witnesses inconsistent with the conclusions but not the premises of the inference-forms intuitionism characteristically rejects: paradigmatically, a kind of negation of p (which could be neither intuitionistic nor classical negation) inconsistent with p but not with $\neg\neg p$. Intuitionistic negation provides defeasors only for inferences rejected by *classical* logic. What stands to intuitionistic negation in this respect as it stands to classical negation (which of course is already in equilibrium in the sense of being defeasor-complete)?

Notice that nothing in this discussion of relations between the consequence relations of intuitionistic and classical logic requires the appeal to notions of <u>truth</u>, or <u>truth-value</u>, or <u>bivalence</u>. The difference in the contribution of the two different sorts of negation to the consequence relation is adequately characterized entirely in terms of the notion of logical incompatibility, in the form of inconsistency, that they codify. From the point of view of the pragmatic expressive approach to the demarcation of logical vocabulary pursued here, understanding those negations is a matter of understanding which aspects of *material* incompatibility they make explicit.

2.4 Relevance logic Relevance logic claims as its primary, in some sense characteristic, virtue its *rejection* of the principle *ex falso quodlibet*. It is just this principle on which the construction offered here of standard incompatibility relations from standard consequence relations turns. For the incoherent sets (and hence the incompatibility relations between sets) are defined as those that count *everything* among their consequences. Now, there is nothing sacred or inevitable about this procedure. What is needed to get the enterprise off the ground is *some* way of picking out sets of sentences that are incoherent, in terms of their distinctive role in the consequence relation. In the present case, where what is at issue is a *logical* consequence relation, incoherence amounts to inconsistency. So the question that must be addressed to relevance logic is: What property, expressible entirely in terms of the consequence relation, distinguishes *inconsistent* from *consistent* sets of sentences? It will *not* be, of course, that the inconsistent ones are those that have *everything* as their consequences. But if not that property, what does distinguish the inconsistent sets in the context of the consequence relation of relevance logic? The unsettling answer, at least for the pure arrow fragment of R and E—the philosophical core and paradigm of the enterprise—is: nothing. There is no way at all to distinguish inconsistent from consistent sets purely on the basis of their role in the consequence relation. (Of course, one can always pick them out as the sets that contain or entail both *p* and *not-p*. But the point is that one cannot recover *that* information just from how the sets behave as premises or conclusions of logically good inferences.)

Now, even if there *were* some way of picking out the relevantly inconsistent sets, it would still have to be shown, and might not be true, that, when treated as the incoherent sets, they would define a *standard* incompatibility relation. (It's a tough counterfactual.) But one might be given pause by the fact that as far as the consequence relation of relevance logic is concerned, consistent and inconsistent sets are indistinguishable. That seems like a Bad Thing, suggesting that this way of recoiling from *ex falso quodlibet* has somehow gone too far. It is a defect that can be remedied. Various "impure" forms of relevance logic introduce special notions of and signs for *absurdity*—none of which, of course, automatically have the consequence of entailing everything—which do permit the discrimination of inconsistent sets as those that have absurdity among their consequences. I've not looked at those to see which, if any, might generate standard incompatibility relations.

This observation about the peculiar inability of relevance logic to discriminate inconsistent sets of sentences solely on the basis of their behavior as premises or conclusions of logically good inferences raises the general question of what one *can* tell about the logical form of sentences in virtue of their role in logical consequence relations, in the context of different logics. It is easy to show that, under natural assumptions, it is possible to recover the logical form of individual sentences from the role they play as premises and conclusions in the *classical* logical consequence relation, *up to negation*. That is, first, because p is logically equivalent to $\sim\sim p$, one cannot tell sentences of those two forms apart. And further, one cannot in general tell p and $\sim p$ apart—not in the sense that they have the *same* role in the consequence relation (after all, $p \vdash p$ and not $p \vdash \sim p$), but in the sense that systematically substituting p for $\sim p$, or vice versa, makes no difference to the consequence relation. On the other hand, in the *intuitionistic* setting, one can not only tell p from $\sim\sim p$, there is also a systematic asymmetry between the consequences and consequential antecedents of p and $\sim p$. The intuitionistic logical consequence relation is *categorical* for the logical form of the sentences it relates, in the sense that it suffices fully to determine their logical form. From an inferentialist semantic perspective, this feature amounts to a significant expressive advantage of intuitionist over classical (not to mention relevance) logic.

Queries in Chapter 5

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